Katholieke Universiteit Leuven

Faculteit Wetenschappen

Campus Kortrijk

The Anosov relation for Nielsen numbers of maps of infra-nilmanifolds

Promotoren:
Prof. dr. K. Dekimpe
Prof. dr. W. Malfait

Proefschrift voorgedragen tot het behalen van de graad van Doctor in de Wetenschappen door

Bram DE ROCK

2006

Dankwoord

Op dit euforisch moment zou ik me voor één keer in mijn leven willen vergelijken met Tom Boonen. Niet dat ik wil beweren dat ik even getalenteerd zou zijn, maar enkel op basis van zijn talent kan Tom geen koers winnen. Hetzelfde geldt natuurlijk voor het schrijven van een doctoraatsthesis en daarom zou ik hier enkele mensen willen bedanken.

Vooreerst kwam ik zes jaar geleden door de kordate aanpak van 'manager' Paul terecht in een schitterende ploeg. Bedankt Paul voor de vaderlijke steun die me steeds voortstuwde.
In elke succesvolle ploeg is er nood aan een 'wegkapitein', wat ook bij mij niet anders was. Bedankt Wim dat je me door de eerste moeilijke jaren loodste, me binnengeleid hebt in de wereld van de vaste punten en dat je er bij was tot op het einde van de rit. Ik weet dat dit niet altijd evident geweest moet zijn.
De juiste wielerterm vinden voor Karel is niet evident. Ietwat oneerbiedig zou ik hem 'meesterknecht' kunnen noemen. Op elk moment stond hij klaar om me te helpen bij één van mijn vele problemen of om me op weg te zetten naar nieuwe resultaten. Bovendien nam hijdirect de kop over als Wim een stap opzij zette. Bedankt Karel dat je me in een zetel naar de finish bracht.

In een koers zijn er echter verschillende ploegen aanwezig en ik had het geluk tot twee ploegen te mogen behoren. Ik wil de economisten, en in het bijzonder Laurens en André, bedanken dat ze een wiskundige in hun midden tolereerden en zelfs overtuigden om zijn wiskundige kennis toe te passen in een economisch kader. Jullie zorg(d)en voor een zeer aangename werksfeer.

Dit moment van euforie zou veel van zijn uitstraling verliezen mocht ik het niet kunnen delen met de vele supporters. Jullie stonden langs de kant van de weg steeds klaar met een bemoedigend woord, een duwtje in de rug, een ontspannend moment om die wiskunde even te kunnen vergeten,.... Kortom, jullie geloofden rotsvast in de goede afloop en zorgden voor de ideale omstandigheden. Bedankt aan jullie allen en in het bijzonder aan mijn ouders en familie, Pastoor Michiel, Dries, Bram en David.

Mijn laatste dankbetuigingen gaan naar Michelle. Het was/is niet altijd evident om met een wiskundige samen te leven. Bedankt voor je vele geduld en de ruimte die je me liet om dit werk te beëindigen. Maar heel in het bijzonder wil ik je danken voor Daan die me -wat soms wel eens nodig is- als geen ander kan kalmeren met zijn ondeugende glimlach.

Bram

Preface

In this dissertation we examine the 'Anosov relation' for (continuous) self-maps $f : M \to M$ of infra-nilmanifolds M. A map f satisfies the Anosov relation if the Nielsen number $N(f)$ and the Lefschetz number $L(f)$ are equal up to sign, or equivalently if $N(f) = |L(f)|$. These numbers find their origin in fixed point theory and they provide information on the fixed points of the map f. More precisely, one of the main objectives of fixed point theory is to calculate $MF(f)$ which is the minimum number of fixed points of all maps homotopic to f. This number however is not readily computable from its definition and therefore the lower bound $N(f)$ is introduced. F. Wecken and B. Jiang showed that under specific conditions, which infra-nilmanifolds satisfy, this lower bound is sharp, i.e. $MF(f) = N(f)$ (see [30],[53]). But again the Nielsen number is not readily computable from its definition and so one can to try to calculate it in another way. One possibility is to use another number associated to a map f, namely the Lefschetz number $L(f)$. In 1985 D. Anosov proved in [1] the following nice theorem.

Theorem 0.1. *For any (continuous) self-map $f : M \to M$ of a nilmanifold M we have that $N(f) = |L(f)|$.*

This is a very interesting result since $L(f)$ is, in opposite to the previously introduced numbers, computable from its definition. So for self-maps f of nilmanifolds, D. Anosov showed that one can calculate $MF(f)$. In order not to be disrespectful to the authors of [2], we want to note that this result of D. Anosov is actually a generalization of their result on the tori.

Since D. Anosov's theorem is only valid for a specific class of manifolds, a natural question is of course whether this theorem can again be generalized towards other classes of manifolds. In this respect, D. Anosov himself already showed that his theorem does not hold on the Klein bottle. The goal of this dissertation is to verify whether the Anosov theorem can be generalized towards (certain classes of) infra-nilmanifolds or more generally to verify whether the Anosov relation holds for a specific map $f : M \to M$ of an infra-nilmanifold M.

The generalization towards infra-nilmanifolds is quite natural, since nilmanifolds and the Klein bottle are both examples of infra-nilmanifolds. Moreover infra-nilmanifolds are a well studied class of manifolds which, amongst having other properties, can be studied via a nice algebraic description based on almost-crystallographic groups. This description will turn out to be crucial in the proofs of our results and therefore we recall the main properties and definitions of infra-nilmanifolds in a first chapter.
In this chapter we also show that in general infra-nilmanifolds and solvmanifolds are not the same. Note that solvmanifolds are also a natural generalization since the nilmanifolds and the Klein bottle can also be seen as solvmanifolds. Therefore it is important to note that the techniques developed for solvmanifolds, as in e.g. [27], [28], [31], can not be straightforwardly applied to infra-nilmanifolds.
In a second preliminary chapter we recall the definitions and properties of the numbers (associated to a map) mentioned above. We start with a very general description and then we apply these concepts to maps of infra-nilmanifolds. It turns out that these concepts are well understood for infra-nilmanifolds. Moreover K.B. Lee proved in [38] a nice criterion to verify the Anosov relation for a given map of an infra-nilmanifold. This result is essential for this dissertation and is therefore also introduced in the second chapter.

The second part of this dissertation consists of our extensions of the Anosov theorem. There are actually two possible ways of trying to generalize the Anosov theorem. A first (natural) way is to look for classes of manifolds, other than nilmanifolds, for which the relation holds for all continuous maps of the given manifold. For instance E. Keppelmann and C. McCord established this for exponential solvmanifolds ([31]).

In this dissertation we introduce several classes of infra-nilmanifolds for which the Anosov theorem holds. All these classes are defined by means of the holonomy group of an infra-nilmanifold. As we will explain in detail in the first chapter, the fundamental group of an infra-nilmanifold is a torsion-free almost-crystallographic group (called almost-Bieberbach group) and to such groups one associates a uniquely determined finite group called the holonomy group. For instance the holonomy group of the Klein bottle is $\mathbb{Z}_2$.

The first class of infra-nilmanifolds for which we obtain an extension of the Anosov theorem, are the infra-nilmanifolds with an odd order holonomy group. This is examined in Chapter 3. In order to show that the Anosov theorem holds for these manifolds, we introduce the very useful concept of a periodic sequence associated to a map.

The Klein bottle is not an element of this first class of manifolds, however it is an example of an infra-nilmanifold with a cyclic holonomy group, the easiest class of finite groups. In Chapter 5 we take a closer look to this class of infra-nilmanifolds having a cyclic holonomy group. Because of the observations of D. Anosov we already know that the Anosov theorem certainly does not hold for the whole class. Therefore we have to introduce extra conditions in order to be able to generalize the Anosov theorem. To find this extra condition, we consider the holonomy representation, which is a faithful matrix representation of the holonomy group, determined by the given infra-nilmanifold (the almost-Bieberbach group). To be more precise, since the holonomy group is cyclic we can assume that is generated by x_0 and by using the holonomy representation, we can associate a matrix A to x_0 (see again the first chapter for an exact description). Then, in case -1 is not an eigenvalue of A, we are able to generalize the Anosov theorem towards the given infra-nilmanifold.
In this way we obtain a sufficient condition for infra-nilmanifolds with a cyclic holonomy group for the Anosov theorem to hold. Note that the Klein bottle, or any other infra-nilmanifold with $\mathbb{Z}_2$ as its holonomy group, does not satisfy this condition, since in that case -1 is an eigenvalue of the associated matrix.
Moreover, for flat manifolds we are able to show that this sufficient condition is sharp, i.e. it is also a necessary condition. This means that for flat manifolds with a cyclic holonomy group, we obtained a complete picture. Unfortunately this is not the case for infra-nilmanifolds, since we constructed an example of an infra-nilmanifold with $\mathbb{Z}_2$ as holonomy group (and so the above condition is not satisfied) for which the Anosov theorem does hold. In fact, we introduced this example already in Chapter 3 since, as we will argue below, it is already useful in that chapter. This example nicely shows that the validity of the

Anosov theorem is much more delicate for infra-nilmanifolds than for flat manifolds.

A last class of infra-nilmanifolds for which we are able to prove that the Anosov theorem holds, is the class of the flat orientable generalized Hantzsche-Wendt manifolds. This is a very specific, well studied class of infra-nilmanifolds which, as one might expect, generalizes the classical 3-dimensional Hantzsche-Wendt manifold. Namely, these manifolds are orientable n-dimensional flat manifolds having $\mathbb{Z}_2^{n-1}$ as their holonomy group. Note that although the Klein bottle is a 2-dimensional flat manifold with $\mathbb{Z}_2$ as holonomy group, it is not an element of this class, since the Klein bottle is non-orientable.
The reason for studying this class, lies in the fact that it is the complete opposite of the classes we worked with thus far. To explain this, let's consider again the matrices occurring in the holonomy representation. So for any element x_i of the holonomy group we have an associated matrix A_i. In our first class of infra-nilmanifolds, the condition of being of odd order boils down to the fact that for any A_i, -1 is not an eigenvalue. For the second class, we required an analogous condition, but now only for the matrix associated to the generator of the cyclic holonomy group. It's perhaps important to remark here that -1 can be an eigenvalue of the other elements (non generators) of the holonomy group and this makes things much more complicated.
In contrast to this, we have that for the flat orientable generalized Hantzsche-Wendt manifolds of dimension n, any A_i will have -1 as an eigenvalue (and for many elements even of multiplicity $n-1$). Nevertheless in Chapter 6 we are still able to show that the Anosov theorem does hold for this class of manifolds.

A second different approach to generalize the Anosov theorem is to search for classes of maps on a certain (type of) manifold, for which the theorem holds. For instance, S. Kwasik and K.B. Lee proved in [34] that the Anosov theorem holds for homotopic periodic maps of infra-nilmanifolds and C. McCord extended this to homotopic periodic maps of infra-solvmanifolds ([45]). Other examples are the virtual unipotent maps introduced and studied in [42] by W. Malfait.

We also followed this approach in this dissertation. In Chapter 3, we examine expanding maps of infra-nilmanifolds and we introduce the nowhere expanding maps of infra-nilmanifolds which are, as one may expect, the antipole of the expanding maps. Again, by using the concept of a periodic sequence associated to a map, we are able to show that the Anosov theorem can be generalized for the latter class of maps. For the expanding maps we obtain a necessary and sufficient condition. To be precise, the Anosov relation holds for a given expanding map $f : M \to M$ of an infra-nilmanifold M if and only if M is orientable.

This last result can be used in order to try to show that the Anosov theorem never holds for non-orientable infra-nilmanifolds. However, this trick does not work on any infra-nilmanifold since there are (non-orientable) infra-nilmanifolds which do not admit expanding maps. The example mentioned before is an example of such an infra-nilmanifold and is therefore presented in Chapter 3. Finally it is important to note that every flat and 2-step nilpotent infra-nilmanifold admit expanding maps which implies that the Anosov theorem never holds for such infra-nilmanifolds if they are non-orientable. A result which will appear to be very useful in Part 3.

In Chapter 4 we examine a third well studied class of maps, namely the Anosov diffeomorphisms. It is well known that not every infra-nilmanifold admits such a map. Up till now, only for the flat manifolds a complete description of which of them admitting an Anosov diffeomorphism is known (see [49]). Therefore we concentrate on flat manifolds. In [41], it is proven that if a flat manifold M admits Anosov diffeomorphisms, then its first Betti number $b_1(M)$ satisfies one of the following: $b_1(M) = 0$, $2 \leq b_1(M) \leq n - 2$ or $b_1(M) = n$ (and all situations occur). In the last case M is a torus and so $N(f) = |L(f)|$ for every continuous self-map f of M. For the other cases, we investigate the possibility of constructing a flat manifold M, with prescribed first Betti number, such that on the one hand M admits an Anosov diffeomorphism f with $N(f) \neq |L(f)|$ and on the other hand M also supports an Anosov diffeomorphism g satisfying $N(g) = |L(g)|$. This is almost always possible except for some very restrictive cases. Namely, for primitive flat manifolds M, i.e $b_1(M) = 0$, in dimension 6 we obtain that the Anosov theorem holds for Anosov diffeomorphisms. On the other hand for n-dimensional flat manifolds M with $b_1(M) = n - 2$ we obtain that the Anosov relation never holds for Anosov diffeomorphisms.

In Part 3 of this dissertation we focus on the infra-nilmanifolds in low dimensions, i.e. up to dimension 4. It appears that many of these infra-nilmanifolds are already covered by our results obtained in the second part (or Anosov's original result). However there are still manifolds which are not yet treated, so examining these manifolds is fruitful as a possible source of inspiration for future research.

In order to handle all infra-nilmanifolds up to dimension 4, we need a list of all possible manifolds. For the flat manifolds we use the description of [7] and for the infra-nilmanifolds (with non-abelian universal covering group) we use the classification of [12].

In Chapter 7 we consider the flat manifolds and we start by showing that in dimension 3 we have, by the known theorems from the second part, already a complete picture. In dimension 4 these theorems cover 53 of the 74 manifolds and the remaining flat manifolds are divided into two groups: the ones with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as their holonomy group and the ones with a non-abelian holonomy group. The first group is interesting since it resembles a lot the classes of infra-nilmanifolds we already examined. It turns out that this class provides some nice counter examples to possible questions which arise naturally from the results obtained in the second part. For instance we show that the validity of the Anosov theorem is quite subtle and can not completely be determined from the holonomy representation alone.
For the second group, similar results are obtained in case the flat manifolds have D_8 as their holonomy group. But more interestingly, we find that the Anosov theorem always holds for (4-dimensional) flat manifolds with A_4 as their holonomy group. This leads of course to the question whether or not this holds for any flat manifold with A_4 as holonomy group?

For the infra-nilmanifolds, basically the same calculations need to be done, however there are some complications. Firstly, the flat manifolds are determined by Bieberbach groups, which naturally arise as matrix groups. This matrix representation is no longer automatically provided for the almost-Bieberbach groups, which complicates our calculations. Secondly, the fact that we work with non-abelian universal covering groups puts extra restrictions on the (matrices describing the) endomorphisms, and hence also the affine endomorphisms of the group. These extra restrictions appear to have important consequences for the validity of the Anosov theorem. For instance, we can use them to show that the Anosov theorem always holds for 4-dimensional infra-nilmanifolds with a non-abelian holonomy group. This is a rather surprising result when we compare this to the corresponding situation for flat manifolds.

This leads of course to the question how the structure of the universal covering group influences the validity of the Anosov theorem?

In Part 3 we present for each (almost-)Bieberbach group a proof of, or a counter example to, the Anosov relation for this specific (almost-) Bieberbach group (or infra-nilmanifold). As we already argued above, an analysis of the obtained results can hopefully form the basis for future research.

Bram De Rock
Kortrijk, February 2006

Contents

Part III The Anosov theorem in small dimensions

Part IV Nederlandstalige samenvatting

Part I

Preliminaries

Chapter 1

Maps of infra-nilmanifolds: an algebraic description

In this dissertation we examine the Anosov relation for maps of infra-nilmanifolds. Therefore we start by giving basic definitions and properties concerning maps of infra-nilmanifolds. In a second chapter we introduce the Anosov relation.

Infra-nilmanifolds are a large, well studied class of manifolds which can be nicely described algebraically. To be specific, an infra-nilmanifold is covered by a nilpotent Lie group and is completely determined by its fundamental group which is an almost-Bieberbach group. Our results formulated in part two are obtained by using specific properties and classes of almost-Bieberbach groups. Therefore the first two sections of this chapter contain definitions and properties concerning Lie groups and infra-nilmanifolds.

In the third section we introduce a result of K.B. Lee ([38]) which gives, up to homotopy, a nice algebraic description of maps of infra-nilmanifolds. This is very convenient since the Anosov relation is homotopy invariant.

Because of these algebraic properties of maps of infra-nilmanifolds and the manifolds themselves, we are able to translate the 'topological' problem into an 'algebraic' one. This translation is crucial and therefore we illustrate carefully all the related concepts by means of examples. [1]

1.1 Lie groups

We recall some basic facts concerning nilpotent Lie groups and refer to [40], [50] and [52] for a more detailed study.

[1] Most of the examples introduced in Part I can also be found in [17].

Throughout this work G is a connected, simply connected, nilpotent Lie group. An affine endomorphism of G is an element (g,φ) of the semigroup $G\rtimes\mathrm{Endo}(G)$ with $g\in G$ the translational part and $\varphi\in\mathrm{Endo}(G)$ (= the semigroup of all continuous endomorphisms of G) the linear part. The product of two affine endomorphisms is given by $(g,\varphi)(h,\mu)=(g\cdot\varphi(h),\varphi\mu)$ and (g,φ) maps an element $x\in G$ to $g\cdot\varphi(x)$. If the linear part φ belongs to $\mathrm{Aut}(G)$, then (g,φ) is an invertible affine transformation of G. We write $\mathrm{Aff}(G)=G\rtimes\mathrm{Aut}(G)$ for the group of invertible affine transformations of G.

Example 1.1. *The best known example of a connected, simply connected, non-abelian Lie group is the Heisenberg group*

$$H=\{\begin{pmatrix}1&y&z\\0&1&x\\0&0&1\end{pmatrix}\mid x,y,z\in\mathbb{R}\}.$$

For further use, we will use $h(x,y,z)$ to denote the element $\begin{pmatrix}1&y&\frac{1}{3}z\\0&1&x\\0&0&1\end{pmatrix}$. (The reason for introducing a 3 in the upper right corner lies in the use of this example later on). One easily computes that

$$h(x_1,y_1,z_1)h(x_2,y_2,z_2)=h(x_1+x_2,y_1+y_2,z_1+z_2+3x_2y_1).$$

Let us fix the following elements for use throughout this chapter: $a=h(1,0,0)$, $b=h(0,1,0)$ and $c=h(0,0,1)$. The group N generated by the elements a,b,c has a presentation of the form

$$N=\langle a,b,c\mid [b,a]=c^3,\ [c,a]=[c,b]=1\rangle.$$

(We use the convention that $[b,a]=b^{-1}a^{-1}ba$.) Obviously the group N consists exactly of all elements $h(x,y,z)$, for which $x,y,z\in\mathbb{Z}$.

For any connected, simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, it is known that the exponential map $\exp:\mathfrak{g}\to G$ is bijective and we denote by log the inverse of exp.

Example 1.2. *The Lie algebra of H is the Lie algebra of matrices of the form*

$$\mathfrak{h}=\{\begin{pmatrix}0&y&z\\0&0&x\\0&0&0\end{pmatrix}\mid x,y,z\in\mathbb{R}\}.$$

The exponential map is given by

$$\exp : \mathfrak{h} \to H : \begin{pmatrix} 0 & y & z \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & y & z + \frac{xy}{2} \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\log : H \to \mathfrak{h} : \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & y & z - \frac{xy}{2} \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$

For later use, we fix the following basis of $\mathfrak{h}$*:*

$$C = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \log(c),\ B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \log(b),$$

$$\textit{and } A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \log(a).$$

For any endomorphism φ of the Lie group G to itself there exists an unique endomorphism φ_* of the Lie algebra $\mathfrak{g}$ (namely the differential $d\varphi$ of φ), making the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, shift left, "\log"] & G \arrow[d, shift left, "\log"] \\ \mathfrak{g} \arrow[r, "\varphi_*"'] \arrow[u, shift left, "\exp"] & \mathfrak{g} \arrow[u, shift left, "\exp"] \end{tikzcd}$$

Conversely, every endomorphism φ_* of $\mathfrak{g}$ appears as the differential of an endomorphism of G.

Example 1.3. *Let H and $\mathfrak{h}$ be as before. With respect to the basis C, B and A (in this order!), any endomorphism φ_* is given by a matrix of the form*

$$\begin{pmatrix} k_1 l_2 - k_2 l_1 & l_3 & k_3 \\ 0 & l_2 & k_2 \\ 0 & l_1 & k_1 \end{pmatrix}.$$

This follows from the fact that $3C = [B, A]$ and hence $3\varphi_*(C) = [\varphi_*(B), \varphi_*(A)]$. Conversely, any such a matrix represents an endomorphism of $\mathfrak{g}$. The corresponding endomorphism φ of H satisfies

$$\begin{aligned}
&\varphi(h(x,y,z))\\
&= \exp(\varphi_*(\log(h(x,y,z))))\\
&= h(k_1x+l_1y, k_2x+l_2y,\\
&\quad 3k_3x+3l_3y+\frac{3(k_1x+l_1y)(k_2x+l_2y)}{2}+(k_1l_2-k_2l_1)(z-\frac{3xy}{2})).
\end{aligned}$$

As one sees, although the map φ_* is linear and thus easy to describe, the corresponding φ is much more complicated. In order to be able to continue presenting examples, we will use a matrix representation of the semigroup $H \rtimes \mathrm{Endo}(H)$ introduced in the previous examples. Given an endomorphism φ of H, let us denote by M_φ the (4×4)-matrix

$$M_\varphi = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$$

where P denotes the (3×3)-matrix, representing φ_* with respect to the basis C, B, A (again in this fixed order). Define the map

$$\begin{aligned}
&\psi : H \rtimes \mathrm{Endo}(H) \to M_4(\mathbb{R}) :\\
&(h(x,y,z),\varphi) \mapsto \begin{pmatrix} 1 & \frac{-3x}{2} & \frac{3y}{2} & \frac{-3xy}{2}+z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot M_\varphi.
\end{aligned} \tag{1.1}$$

One easily verifies that ψ defines a faithful representation of the semigroup $H \rtimes \mathrm{Endo}(H)$ into the semigroup $M_4(\mathbb{R})$ (respectively of the group $\mathrm{Aff}(H)$ into the group $\mathrm{Gl}(4,\mathbb{R})$).

Remark 1.4. *An analogous matrix representation can be obtained for any $G \rtimes \mathrm{Endo}(G)$ in case G is two-step nilpotent. (Recall that a group G is said to be k-step nilpotent if the $(k+1)$'th term of the lower central series $\gamma_{k+1}(G) = 1$, where $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. E.g. the Heisenberg group is 2-step nilpotent.) This is proved in [11] for the group $\mathrm{Aff}(G)$, but the details in that paper can easily be adjusted to the case of the semigroup $G \rtimes \mathrm{Endo}(G)$.*

1.2 Infra-nilmanifolds

When we use infra-nilmanifolds we will always refer to the fundamental group which is an almost-Bieberbach group. In this section we give a short introduction to infra-nilmanifolds and some related concepts. We refer to [8] and [12] for more details.

An almost-crystallographic group is a subgroup E of $\mathrm{Aff}(G)$ such that its subgroup of pure translations $N = E \cap G$, is a uniform lattice (by which we mean a discrete and cocompact subgroup) of G and moreover, N is of finite index in E. Therefore the quotient group $F = E/N$ is finite and is called the holonomy group of E. This implies that E fits into the following short exact sequence:

$$1 \to N \to E \to F \to 1$$

Note that the group F is isomorphic to the image of E under the natural projection $\mathrm{Aff}(G) \to \mathrm{Aut}(G)$, and hence F can be viewed as a subgroup of $\mathrm{Aut}(G)$ and of $\mathrm{Aff}(G)$.

Any almost-crystallographic group acts properly discontinuously on (the corresponding) G and the orbit space $E\backslash G$ is compact. Recall that an action of a group E on a locally compact space X is said to be properly discontinuous if for every compact subset C of X, the set $\{\gamma \in E \mid \gamma C \cap C \neq \emptyset\}$ is finite. When E is a torsion-free almost-crystallographic group, it is referred to as an almost-Bieberbach group and the orbit space $M = E\backslash G$ is called an infra-nilmanifold. M is called a nilmanifold in the special case that $E = N$ (and consequently $F = 1$). For (infra-)nilmanifolds E equals the fundamental group $\pi_1(M)$ of the infra-nilmanifold M and we will also talk about F as being the holonomy group of M. When we refer in the sequel to the universal covering group G of an infra-nilmanifold, we actually refer to the above construction.

Any almost-crystallographic group determines a faithful representation $T : F \to \mathrm{Aut}(G)$, which is induced by the natural projection $p : \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$, and which is referred to as the holonomy representation.

Remark 1.5. *As isomorphic almost-crystallographic subgroups are conjugated inside* $\mathrm{Aff}(G)$ *(see Theorem 1.10 below or [39]), it follows that the holonomy representation of an almost–crystallographic group is completely determined from the algebraic structure of E up to conjugation by an element of* $\mathrm{Aff}(G)$.

Let $\mathfrak{g}$ denote the Lie algebra of G. By taking differentials, the holonomy representation also induces a faithful representation

$$T_* : F \to \mathrm{Aut}(\mathfrak{g}) : x \mapsto T_*(x) = d(T(x)).$$

This faithful representation will be crucial in the proofs of the results in the second part. As a first application we introduce the following very convenient condition concerning the orientability of an infra-nilmanifold.

Proposition 1.6. *Let M be an infra-nilmanifold with holonomy group F and associated holonomy representation $T : F \to \mathrm{Aut}(G)$. Then*

- *M is orientable $\Leftrightarrow \forall x \in F : \det(T_*(x)) = 1$*
- *M is non-orientable $\Leftrightarrow \exists x \in F : \det(T_*(x)) = -1$*

For more background concerning this proposition see [4, page 211] and [12, page 135].

Example 1.7.
Let φ be the automorphism of H, whose differential φ_* is given by the matrix $\begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Let $\alpha = (h(0,0,\frac{1}{3}), \varphi) \in \mathrm{Aff}(H)$. Then the group E generated by a, b, c and α has a presentation of the form

$$E = \langle a, b, c, \alpha \mid \begin{array}{llll} [b,a] = c^3 & [c,a] = 1 & [c,b] = 1 & \\ \alpha a = b\alpha & \alpha b = a^{-1}b^{-1}\alpha & \alpha c = c\alpha & \alpha^3 = c \end{array} \rangle$$

(this is easily checked using the matrix representation (1.1)). E is an almost-crystallographic group with translation subgroup $N = H \cap E = \langle a, b, c\rangle$ and a holonomy group $F = E/N \cong \mathbb{Z}_3$ of order three (see also [12, page 164, type 13]). We have that

$$T_*(F) = \left\{ I_3, \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \right\}$$

(of course I_n denotes the $(n \times n)$-identity matrix).
As E is torsion-free, it is an almost-Bieberbach group and it determines an infra-nilmanifold $M = E\backslash H$. M is orientable since clearly $\det(T_*(x)) = 1$ for all $x \in F$.

In the special case that G is abelian, i.e. $G = \mathbb{R}^n$, M is a flat manifold. In this case the fundamental group $\pi_1(M)$ is called a torsion-free crystallographic group or Bieberbach group. The fundamental group always fits into an extension

$$1 \to \mathbb{Z}^n \to \pi_1(M) \to F \to 1$$

with F the holonomy group and n the dimension of the manifold. Most of the concepts introduced above are easily adapted but it is worth mentioning that in this case $T_*(x) = T(x)$ since the holonomy representation can also be seen as a map from F to $\mathrm{Gl}(n, \mathbb{Z})$. This simplifies a lot the calculations (and notations), as will be demonstrated later on.

Finally we would like to use the same example once more to show that infra-nilmanifolds and solvmanifolds are different even if at first sight they do not seem so. Although the fundamental group of an infra-nilmanifold with an abelian holonomy group is always solvable (in fact polycyclic), these manifolds do not need to be solvmanifolds in general and so the Nielsen theory on these manifolds cannot be treated by the techniques developed for solvmanifolds (as in e.g. [27], [28], [31]).

Example 1.8. *The almost-Bieberbach group $E = \langle a, b, c, \alpha \rangle$ of Example 1.7 is not the fundamental group of a solvmanifold. Indeed, suppose that E is the fundamental group of a solvmanifold, then it is known that the manifold must admit a Mostow fibering over a torus with a nilmanifold as fibre. On the level of the fundamental group, this implies that there exists a short exact sequence*

$$1 \to \Gamma \to E \to A \to 1 \tag{1.2}$$

where Γ is a finitely generated torsion-free nilpotent group and A is a free abelian group of finite rank. However, it is easy to see that $[E, E]$ is of finite index in E, and therefore, the only free abelian quotient of E is the trivial group. Therefore, there does not exist a normal nilpotent group $\Gamma \subseteq E$, with E/Γ free abelian. This shows that E is not the fundamental group of a solvmanifold.

Remark 1.9. *To be precise: a group E which admits such a short exact sequence as introduced in 1.2 is called a strongly torsion-free S group, with S a solvable Lie group. Moreover E is the fundamental group of a solvmanifold if and only if E is a strongly torsion-free S group. Note that this condition implies that E is solvable but not vice versa. We refer to [44] for more details.*

1.3 Maps of infra-nilmanifolds

In the previous section we saw that infra-nilmanifolds are completely determined by an almost-Bieberbach group. To optimally use this algebraic description we need analogous results for (continuous) maps of infra-nilmanifolds. These are provided by K.B. Lee in [38].

He proved the following theorem concerning almost-crystallographic groups.

Theorem 1.10. *Let $E, E' \subset \mathrm{Aff}(G)$ be two almost-crystallographic groups. Then for any homomorphism $\theta : E \to E'$, there exists a $g = (\delta, \mathfrak{D}) \in G \rtimes \mathrm{Endo}(G)$ such that $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in E$.*

Important for us is the following corollary of this theorem (we refer to [38] for a detailed proof).

Corollary 1.11. *Let $M = E\backslash G$ be an infra-nilmanifold and $f : M \to M$ a continuous map of M. Then f is homotopic to a map $h : M \to M$ induced by an affine endomorphism $(\delta, \mathfrak{D}) : G \to G$.*

We say that $(\delta, \mathfrak{D})$ is a homotopy lift of f. Note that one can find the homotopy lift of a given f, by using Theorem 1.10 for the homomorphism $f_* : \pi_1(M) \to \pi_1(M)$ induced by f. In fact, using this method one can characterize all continuous maps, up to homotopy, of a given infra-nilmanifold M. Indeed, since E is finitely generated, one can construct all possible homomorphisms $\theta : E \to E$ and so, by Theorem 1.10, all possible homotopy lifts $(\delta, \mathfrak{D})$ of f.

Example 1.12. *Let E be the almost-Bieberbach group of Example 1.7, then there is a homomorphism $\theta_1 : E \to E$, which is determined by the images of the generators as follows:*

$$\theta_1(a) = b^2c^3,\ \theta_1(b) = a^2c^3,\ \theta_1(c) = c^{-4},\ \textit{and}\ \theta_1(\alpha) = c^{-2}\alpha^2.$$

Using the matrix representation (1.1) it is easy to check that θ_1 really determines an endomorphism of E and that this endomorphism is induced by the affine endomorphism $(h(0,0,0), \mathfrak{D})$, where

$$\mathfrak{D}_* = \begin{pmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

There is a second corollary of this theorem which we also want to point out.

Corollary 1.13. *Let $M = E\backslash G$ be an infra-nilmanifold with holonomy group F and $T : F \to \mathrm{Aut}(G)$ the associated holonomy representation. Suppose $f : M \to M$ is a continuous map of M and $(\delta, \mathfrak{D})$ is a homotopy lift of f. Then*

$$\forall x \in F, \exists y \in F : T_*(y)\mathfrak{D}_* = \mathfrak{D}_* T_*(x).$$

Proof: If $\tilde{x} \in E = \pi_1(M)$ is a pre-image of x, then y can be taken as the natural projection of $f_*(\tilde{x})$, where f_* denotes the morphism induced by f on $\pi_1(M)$. □

Let us again demonstrate this with an example.

Example 1.14. *Let $M = E\backslash H$ be the infra-nilmanifold introduced in Example 1.7 and suppose that $f_1 : M \to M$ is a continuous map inducing the endomorphism θ_1 on $E = \pi_1(M)$. We know already that $f_* = \theta_1$ is induced by $(1, \mathfrak{D})$ and it is easy to check that*

$$\mathfrak{D}_*\varphi_* = \begin{pmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = \varphi_*^2 \mathfrak{D}_*$$

Finally, since Theorem 1.10 is very crucial for our results, we would like to warn the reader that in the paper of K.B. Lee ([38, Theorem 2.2]) there is a slight mistake in the proof of the theorem. However as we will now argue this has no real influence on the result.

In the beginning of the proof, it is claimed that given a continuous map $f : M = E\backslash G \to M = E\backslash G$, inducing a morphism $f_* : E \to E$ on the fundamental group E of M, one finds that, with $N = E \cap G$,

$$\Lambda = N \cap f_*^{-1} f_*(N \cap f_*^{-1}(N))$$

is a normal finite index subgroup of E, with $f_*(\Lambda) \subseteq \Lambda$. This is however incorrect. In fact, in general one can see that the definition of Λ given above implies that $\Lambda = f_*^{-1}(N) \cap N$, which already indicates that something might be wrong with the above claim.

As an example, let $E = K \times \mathbb{Z}^2$, where K is the fundamental group of the Klein bottle, then

$$E = \langle a, b, c, d \parallel ab = ba^{-1},\ c, d \text{ central} \rangle.$$

Of course E is a crystallographic group, with translational part $N = \langle a, b^2, c, d \rangle$. Let $f : M \to M$ be the continuous map on the flat manifold with fundamental group E, inducing the homomorphism $f_* : E \to E$ with

$$f_*(d) = c;\ f_*(c) = b;\ f_*(b) = 1 \text{ and } f_*(a) = 1.$$

Then one easily computes that

- $f_*^{-1}(N) = \{a^k b^l c^{2m} d^n \parallel k, l, m, n \in \mathbb{Z}\}$
- $f_*^{-1}(N) \cap N = \{a^k b^{2l} c^{2m} d^n \parallel k, l, m, n \in \mathbb{Z}\}$
- $f_*(f_*^{-1}(N) \cap N) = \{b^{2m} c^n \parallel m, n \in \mathbb{Z}\}$
- $f_*^{-1}(f_*(f_*^{-1}(N) \cap N)) = \{a^k b^l c^{2m} d^n \parallel k, l, m, n \in \mathbb{Z}\}$
- $\Lambda = f_*^{-1}(f_*(f_*^{-1}(N) \cap N)) \cap N = \{a^k b^{2l} c^{2m} d^n \parallel k, l, m, n \in \mathbb{Z}\}$

So $d \in \Lambda$, but $f_*(d) = c \notin \Lambda$, showing that $f_*(\Lambda) \nsubseteq \Lambda$ as was claimed in [38].

Nevertheless, this is of no real influence for the results of [38]. Indeed, all what is needed is a finite index normal subgroup Λ of E, contained in N and satisfying $f_*(\Lambda) \subseteq \Lambda$. Now, suppose $[E : N] = n$ (which is finite since E is an almost-Bieberbach group) then

$$\Lambda = \langle x^n \parallel x \in E \rangle$$

will be such a group (and this Λ is independent of f_*). Most probably K.B. Lee also noticed this since in [37, Lemma 1.1] he introduced the same group.

Chapter 2

The Anosov relation

In the previous chapter we introduced maps of infra-nilmanifolds. In this chapter we discuss what we exactly want to investigate about these maps, namely the so called Anosov relation.

To a self-map $f : M \to M$ of a manifold M one can associate the Nielsen number $N(f)$ and the Lefschetz number $L(f)$. We then say that the Anosov relation holds for this map if $N(f) = |L(f)|$. In 1985, D. Anosov proved in [1] that this relation holds for any continuous map of a nilmanifold. He also showed that this result does not hold in general for infra-nilmanifolds by constructing a counterexample on the Klein bottle.

As the main topic of this dissertation consists of investigating the Anosov relation for maps of infra-nilmanifolds, we first introduce the necessary general concepts concerning the Nielsen and the Lefschetz number. We also explain why fixed point theorists are interested in these numbers and more specifically in the relation between them. This interest is already demonstrated by several publications inspired by the initial result of D. Anosov: [31], [34], [42], [45],. . . .

After introducing the general setting, we again focus in the second section of this chapter to infra-nilmanifolds. In [38], K.B. Lee proved a nice criterion to decide whether the Anosov relation holds for a map of an infra-nilmanifold.

2.1 Fixed point theory

Let $f : X \to X$ be a map of a compact connected manifold and denote the fixed point set of f by $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$. One of the main objectives in fixed point theory is to calculate $MF(f)$, the minimum number of fixed points among all maps homotopic to f:

$$MF(f) = \min\{\#\mathrm{Fix}(g) \mid g \simeq f\}.$$

Thus, for instance, $MF(f) = 0$ means that there exists a map g homotopic to f such that $g(x) \neq x$ for all $x \in X$.

The motivation for calculating this number finds its origin in the study of dynamical systems. Note that we can create a dynamical system by iterating $f : X \to X$. In the study of such a dynamical system it is very interesting to know how many fixed points the system has and moreover, how many of these fixed points are persistent under small perturbations. This boils exactly down to calculating $MF(f)$.

In principle, to calculate $MF(f)$ it is necessary to examine the fixed point set of every map homotopic to f. To avoid this, several numbers associated to f are defined to provide information on $MF(f)$.

2.1.1 The Lefschetz number

A first number, and perhaps the best known one, is the Lefschetz number $L(f)$.

Definition 2.1. *Let $f : M \to M$ be a map of a compact connected manifold M. Then*

$$L(f) = \sum_i (-1)^i \ \mathrm{Trace}(f_* : H_i(M, \mathbb{Q}) \to H_i(M, \mathbb{Q})).$$

The Lefschetz number $L(f)$ is a (reasonably) computable invariant of f and it is a homotopy invariant. Unfortunately it does not give a lot of information on the fixed points of f. We only know that if $L(f) \neq 0$, then f has at least one fixed point which can not be removed by a homotopy. In general however $L(f)$ does not give exact information about the number of such points. Of course it is sometimes sufficient to observe that a dynamical system has at least one fixed point and in this case $L(f)$ is very useful.

2.1.2 The Nielsen number

We now introduce the Nielsen number $N(f)$ which is an (almost always sharp) lower bound for $MF(f)$, but unfortunately $N(f)$ is not readily computable from its definition. F. Wecken and B. Jiang proved that this lower bound is sharp for all compact, connected manifolds except for surfaces of negative Euler characteristic ([30],[53]). This is interesting for us since infra-nilmanifolds satisfy these conditions. Since $MF(f) = N(f)$ for self-maps f of infra-nilmanifolds, the calculation of the Nielsen number becomes now a main objective.

The definition of $N(f)$ is based on the concept of essential fixed point classes. Fixed point classes can be introduced in two equivalent ways. A first way uses the universal covering space $(\tilde{M}, p)$ of the manifold M. To be more precise, one can divide all possible lifts $\tilde{f}$ of a given map $f : M \to M$ to the universal covering space $\tilde{M}$ into equivalence classes, called lift classes. Two lifts $\tilde{f}$ and $\tilde{f}'$ are said to be equivalent if they are conjugated by an element of the fundamental group $\pi_1(M)$ (viewed as the group of covering transformations of $(\tilde{M}, p)$). The total number of these lift classes (which is a positive integer or ∞) is called the Reidemeister number $R(f)$ and this number is a homotopy invariant.

Let $\langle \tilde{f} \rangle$ denote the lift class containing a lift $\tilde{f}$. One can show that the projection $p(\mathrm{Fix}(\tilde{f}))$ of the fixed point set of this lift, is independent of the chosen representant of the lift class $\langle \tilde{f} \rangle$. Moreover, $p(\mathrm{Fix}(\tilde{f}))$ is a (possibly empty!) subset of $\mathrm{Fix}(f)$. By considering all possible lifts (in fact lift classes) of f, these subsets $p(\mathrm{Fix}(\tilde{f}))$ divide $\mathrm{Fix}(f)$ into the so called fixed point classes of f.

In this way we have associated to each lift class a fixed point class and therefore there are exactly $R(f)$ fixed point classes. Of course not every fixed point class necessarily contains fixed points. In that case, we say that the lift class is associated to an empty fixed point class. Note that this characterization can lead to many 'different' empty fixed point classes. In fact, for a given map f on a closed manifold M, there can be only finitely many non-empty fixed point classes, while $R(f)$ can be infinite. The advantage of this definition of a fixed point class is the characterization of empty fixed point classes, which is needed later on. A possible disadvantage is that we need a detour via the universal covering space.

A second way of defining the concept of a fixed point class avoids the use of the universal covering space. We partition $\mathrm{Fix}(f)$ into equivalence classes, referred to as fixed point classes, by the relation: $x, y \in \mathrm{Fix}(f)$ are f-equivalent if and only if there is a path w from x to y such that w and fw are (rel. endpoints) homotopic. A disadvantage of this definition is that is not clear what is meant by an empty fixed point class, although we need this idea. Indeed, in some occasions fixed point classes can be removed by homotopy, i.e. a nonempty fixed point class of f can become an empty fixed point class of a homotopic map g. (Or equivalently this means that the associated lift class no longer has fixed points.)

It can be shown, at least for non empty fixed point classes, that these two definitions are equivalent. See for instance [29] for more details. Fixed point classes which persist under all homotopies are called essential fixed point classes. Therefore to calculate $MF(f)$ we only have to focus on the essential fixed point classes and we define:

Definition 2.2. *Let $f : X \to X$ be a map of a compact connected manifold, then $N(f) =$ the number of essential fixed point classes of f.*

The problem is that it is very hard to decide whether a class is essential or not. If an explicit homotopy can be given that deforms a fixed point class into an empty class, then by definition this class is inessential. But, to prove that a fixed point class is essential, we must verify that no homotopy eliminates this class. Of course this is much harder to realize.

The solution to this problem is to replace the homotopic-theoretic definition of 'essentialness' by an algebraic approximation. Namely, we associate (when possible) an integer index to each fixed point class and define a class to be (algebraic) essential if this index is non-zero. Note that these two notions of essentialness must of course be the same for the respective fixed point classes.

For general manifolds it is not easy to define this index. However when we restrict to compact manifolds this is possible in a satisfactory way. Satisfactory means for instance that the index must be homotopy invariant and that the sum of the indices of all fixed point classes must be equal to $L(f)$. This last demand is called the normalization of the index and $L(f)$ is in this setting also called the total algebraic count of fixed points. We will limit ourselves here to an informal approach introduced by B. Brown in [6]. For a mathematically rigorous presentation we refer for instance to [5], [21], [29], [32], ...

Let F be a fixed point class of the map $f : X \to X$ on a compact manifold X. Then there is an open subset U of X containing F such that the closure of U intersects $\mathrm{Fix}(f)$ only in F. For n large enough, we may embed X in an Euclidian space $\mathbb{R}^n$. Consider then for each point $x \in U \backslash F$ the vector in $\mathbb{R}^n$ from x to $f(x)$. Roughly speaking, if all these vectors point in more or less the same direction, we can modify the definition of f on F to move all those points in the direction indicated by the vectors. In fact in this way we produce a map homotopic to f, and identical to f outside U, that has no fixed points on U. Since F can be eliminated in this way, F is said to be inessential. On the other hand, if the vectors do not all point in somewhat the same direction, the vector field on $U \backslash F$ can be thought of as 'winding around' the set F so that it is not possible to modify f by homotopy in order to eliminate F. Consequently, in this case F is essential. In a rigorous development, the amount of 'winding around' of the vector field is measured by an integer, called the index of F.

2.1.3 The Anosov relation

So, we are interested in the Nielsen number because it gives useful information, but unfortunately it is not readily computable from its definition. For the Lefschetz number however, the opposite holds. Therefore we can try to relate the Nielsen number and the Lefschetz number. The best we can hope for is to show that $N(f) = |L(f)|$ since in this case we can compute an almost always sharp lower bound for $MF(f)$. Recall, that for infra-nilmanifolds this lower bound is always sharp. In this context, D. Anosov proved in 1985 the following theorem ([1]):

Theorem 2.3. *For any (continuous) self-map $f : M \to M$ of a nilmanifold M, we have that $N(f) = |L(f)|$.*

He also showed that the theorem does not hold for the Klein bottle by constructing a counterexample. For the reasons mentioned above, the theorem is very convenient and therefore we try to generalize it towards maps of more general manifolds. Since this is the main topic of this dissertation, we want to explicitly define what we examine.

Definition 2.4. *Let M be a closed manifold.*

- *A map $f : M \to M$ satisfies the Anosov relation if and only if $N(f) = |L(f)|$.*
- *M satisfies the Anosov theorem if and only if every map $f : M \to M$ satisfies the Anosov relation.*

In this work we take off from the result of D. Anosov as it concerns infra-nilmanifolds: nilmanifolds and the Klein bottle are infra-nilmanifolds. So it is quite natural to trying generalizing the Anosov theorem towards other classes of infra-nilmanifolds. We have to mention that in literature one already finds several generalizations of this theorem towards (maps of) solvmanifolds (for instance [27], [28],[31]). This is also a natural approach since nilmanifolds and the Klein bottle can also be seen as solvmanifolds. At the end of the first chapter, we clearly demonstrated that the class of solvmanifolds and the class of infra-nilmanifolds do not coincide and therefore the techniques used for solvmanifolds cannot be applied straightforwardly on infra-nilmanifolds (and vice versa).

Finally we want to note that generalizing the Anosov relation is not the only way to try to obtain information on $N(f)$ (and so on $MF(f)$). The Reidemeister number $R(f)$ is another homotopy invariant associated to f and it is also better computable then $N(f)$. From the definitions above it follows that $R(f)$ is an upper bound for $N(f)$, since it also counts the non-essential fixed point classes. However this upper bound can possibly be very bad since $R(f)$ can be infinite. On the other hand a similar relation to the Anosov relation could exist. For instance, B. Norton-Odenthal showed in [48] that $N(f) = R(f)$ for nilmanifolds and P. Wong did the same for a class of manifolds which are totally different from infra-nilmanifolds ([55]). In this dissertation we did not opt to examine this relation, although this could be a possibility for further research.

2.2 Fixed point theory on infra-nilmanifolds

In this section we want to apply the concepts introduced in the previous section to infra-nilmanifolds. This specific setting gives us convenient tools for our research on the Anosov relation. To start with, we note that in [33, Remark 3.7] it is shown that the index of essential fixed point classes of maps on infra-nilmanifolds is always 1 or -1. Although we will not really need this fact, we shortly recall the ideas behind the proof since they give some valuable intuition for the key results mentioned later on. For more details we refer to [33].

Suppose $f : M \to M$ is a map of an infra-nilmanifold M. By a change up to homotopy we may assume that f is induced by (can be lifted to) an affine endomorphism $(\delta, \mathfrak{D})$ of the universal covering Lie group

G of M. By the argumentation at the end of the previous chapter, we know that there exists in $\pi_1(M)$ a normal subgroup of translations, say K, of finite index such that $f_*(K) \subseteq K$. Actually this implies that there exists a nilmanifold N_K which finitely covers M and such that the map f can not only be lifted to the universal covering space, but also to this nilmanifold, say $\bar{f} : N_K \to N_K$. Of course, one of the possible lifts of f to N_K is the map induced by the affine endomorphism $(\delta, \mathfrak{D})$ on the nilmanifold N_K. The other lifts of f to the nilmanifold N_K are induced by an affine endomorphisms $(\delta', \mathfrak{D}') = \alpha(\delta, \mathfrak{D})$ where $\alpha \in \pi_1(M) \subseteq \mathrm{Aff}(G)$.
In [33] it is shown that calculating the index of an essential fixed point class of f boils down to calculating $\frac{L(\bar{f})}{N(\bar{f})}$ for a specific lift $\bar{f} : N_K \to N_K$. This shows that the index of an essential fixed point class is 1 or -1 since $\bar{f}$ is a map of a nilmanifold and so the Anosov relation holds for $\bar{f}$. Finally in the proof of [33, Theorem 3.5] it is shown that $N(\bar{f}) = |\det(I - \mathfrak{D}'_*)|$.

$$\frac{L(\bar{f})}{N(\bar{f})} = \frac{L(\bar{f})}{|L(\bar{f})|} = \mathrm{sgn}(\det(I_n - \mathfrak{D}'_*)) = \pm 1.$$

We gave this description because it illustrates that one has to calculate 'signs of determinants', which of course can result in different signs for different lifts $\bar{f}$. This is completely formalized in the following theorem of K.B. Lee ([38]):

Theorem 2.5. *Let $f : M \to M$ be a continuous map of an infra-nilmanifold M and let $T : F \to \mathrm{Aut}(G)$ be the associated holonomy representation. If $(\delta, \mathfrak{D}) \in G \rtimes \mathrm{Endo}(G)$ is a homotopy lift of f, then*

- *$N(f) = L(f)$ if and only if $\forall x \in F : \det(I_n - T_*(x)\mathfrak{D}_*) \geq 0$.*
- *$N(f) = -L(f)$ if and only if $\forall x \in F : \det(I_n - T_*(x)\mathfrak{D}_*) \leq 0$.*

It can be easily verified that $T(x)\mathfrak{D}$ are exactly all possible linear parts of the homotopy lifts of f. So the result above can be interpreted as follows: consider all possible homotopy lifts of f and verify if the associated fixed point classes have the same index. Therefore this result generalizes indeed the concepts introduced above. For a more detailed study we refer to [38].
The above theorem is crucial for this dissertation because it is a very convenient tool to generalize the Anosov theorem. This will be demonstrated elaborately in the next part. Here we already use it to prove the

following proposition describing a class of maps on infra-nilmanifolds for which the Anosov theorem always holds. Note that we do not claim that such maps exist on all infra-nilmanifolds.

Proposition 2.6. *Let M be an infra-nilmanifold with holonomy group F and associated holonomy representation $T : F \to \mathrm{Aut}(G)$. Let $f : M \to M$ be a continuous map with homotopy lift $(\delta, \mathfrak{D})$. Suppose that $\forall x \in F,\ x \neq 1 :\ T_*(x)\mathfrak{D}_* \neq \mathfrak{D}_* T_*(x)$. Then*

$$\forall x \in F : \det(I_n - \mathfrak{D}_*) = \det(I_n - T_*(x)\mathfrak{D}_*)$$

and hence $N(f) = |L(f)|$.

Proof: Let $x \neq 1 \in F$. Since $(\delta, \mathfrak{D})$ is obtained from Theorem 1.10, Corollary 1.13 implies that there exists an $y \in F$ such that $T_*(y)\mathfrak{D}_* = \mathfrak{D}_* T_*(x)$. Because of the condition on T_* and $\mathfrak{D}_*$ we know that $x \neq y$. Then

$$\begin{aligned}
\det(I_n - \mathfrak{D}_*) &= \det(T_*(x) - \mathfrak{D}_* T_*(x)) \det(T_*(x^{-1})) \\
&= \det(T_*(x^{-1})) \det(T_*(x) - T_*(y)\mathfrak{D}_*) \\
&= \det(I_n - T_*(x^{-1}y)\mathfrak{D}_*).
\end{aligned}$$

Since $x \neq y$ and T_* is faithful, we have that $T_*(x^{-1}y) \neq I_n$. Moreover, for any other $x' \neq 1 \in F$, with $x \neq x'$ and $T_*(y')\mathfrak{D}_* = \mathfrak{D}_* T_*(x')$, we have that $x^{-1}y \neq x'^{-1}y'$. Indeed, suppose that there exists an $x' \in F$, $x \neq x'$, such that $x^{-1}y = x'^{-1}y'$. Then

$$\begin{aligned}
T_*(x^{-1}y)\mathfrak{D}_* = T_*(x'^{-1}y')\mathfrak{D}_* &\Leftrightarrow T_*(x^{-1})\mathfrak{D}_* T_*(x) = T_*(x'^{-1})\mathfrak{D}_* T_*(x') \\
&\Leftrightarrow \mathfrak{D}_* T_*(xx'^{-1}) = T_*(xx'^{-1})\mathfrak{D}_*.
\end{aligned}$$

The last equality is only satisfied when $xx'^{-1} = 1$ which gives us a contradiction. This proves the proposition because any $x \in F$ determines an unique element $x^{-1}y \in F$, and thus all elements of F are obtained. The last conclusion easily follows from Theorem 2.5. □

Remark 2.7. *The condition in this proposition may seem strange, but actually it boils down to the fact that for these maps there is only one Reidemeister class on the covering nilmanifold. In that case, one easily verifies that for such maps the Anosov relation holds. See for instance [29].*

Let us demonstrate this proposition.

Example 2.8. *Let $M = E\backslash H$ be the infra-nilmanifold introduced in Example 1.7 and suppose that $f_1 : M \to M$ is a continuous map inducing the endomorphism θ_1 on $E = \pi_1(M)$. We know already that $f_* = \theta_1$ is induced by $(1, \mathfrak{D})$ and it is easy to check that*

$$\varphi_* \mathfrak{D}_* = \begin{pmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 3 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \mathfrak{D}_* \varphi_*^2$$

which implies that the map f (or $\mathfrak{D}_*$) satisfies the criteria of the proposition. Indeed we have that

$$\det(I_3 - \mathfrak{D}_*) = \det(I_3 - \varphi_* \mathfrak{D}_*) = \det(I_3 - \varphi_*^2 \mathfrak{D}_*) = -15.$$

We end this section with an explicit formula to calculate $L(f)$ and $N(f)$ for self-maps of infra-nilmanifolds established in a recent preprint of J.B. Lee and K.B. Lee ([37]).

Theorem 2.9. *Let M be an infra-nilmanifold with holonomy group F and associated holonomy representation $T : F \to \mathrm{Aut}(G)$. Let $f : M \to M$ be a continuous map with homotopy lift $(\delta, \mathfrak{D})$. Then*

$$L(f) = \frac{1}{|F|} \sum_{x \in F} \det(I_n - T_*(x)\mathfrak{D}_*) \qquad \text{and}$$
$$N(f) = \frac{1}{|F|} \sum_{x \in F} |\det(I_n - T_*(x)\mathfrak{D}_*)|.$$

This is of course a nice generalization of Theorem 2.5. Note that this theorem implies that for maps f of infra-nilmanifolds we always have that

$$MF(f) = N(f) \geq |L(f)|.$$

However this lower bound can be very bad as is demonstrated in the following example.

Example 2.10. *Let E be the Bieberbach group generated by*

$$(e_1, I_5), \cdots (e_5, I_5), (a, A) = (\begin{pmatrix} 1/4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix})$$

where e_i stands for a column vector of length 5 with 1 *in the i-th place and* 0 *everywhere else* ($1 \leq i \leq 5$)*. It is easily verified that E is indeed a Bieberbach group which fits into the following short exact sequence*

$$1 \to \mathbb{Z}^5 \to E \to \mathbb{Z}_4 \to 1$$

So we obtain a 5*-dimensional flat manifold $M = E\backslash\mathbb{R}^5$ with holonomy group $\mathbb{Z}_4$. Then take*

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix})$$

with $k \in 1 + 4\mathbb{Z}$. The affine endomorphism $(\delta, \mathfrak{D})$ induces a map $f : M \to M$ if

$$(\delta, \mathfrak{D})E(\delta, \mathfrak{D})^{-1} = (\delta, \mathfrak{D})E(-\mathfrak{D}^{-1}\delta, \mathfrak{D}^{-1}) \subseteq E$$

This is indeed the fact since for any i, $1 \leq i \leq 5$, we have that

$$\begin{aligned}(\delta, \mathfrak{D})(e_i, I_5)(-\mathfrak{D}^{-1}\delta, \mathfrak{D}^{-1}) &= (\delta + \mathfrak{D}e_i, \mathfrak{D})(-\mathfrak{D}^{-1}\delta, \mathfrak{D}^{-1}) \\ &= (\mathfrak{D}e_i, I_5) \in E\end{aligned}$$

since $\mathfrak{D}e_i \in \mathbb{Z}^5$. Secondly because A and $\mathfrak{D}$ commute we also have

$$\begin{aligned}(\delta, \mathfrak{D})(a, A)(-\mathfrak{D}^{-1}\delta, \mathfrak{D}^{-1}) &= (\delta + \mathfrak{D}a, \mathfrak{D}A)(-\mathfrak{D}^{-1}\delta, \mathfrak{D}^{-1}) \\ &= (\mathfrak{D}a, A) \\ &= ((\mathfrak{D} - I_5)a, I_5)(a, A) \in E\end{aligned}$$

since $(\mathfrak{D} - I_5)a \in \mathbb{Z}^5$. If we then use Theorem 2.9, we obtain

$$\begin{aligned}L(f) &= \frac{1}{4}\sum_{x \in \mathbb{Z}_4} \det(I_5 - T_*(x)\mathfrak{D}_*) \\ &= \frac{1}{4}\sum_{i=0}^{3} \det(I_5 - A^i\mathfrak{D}) \\ &= \frac{1}{4}(0 + 4(1-k) + 8(-1+k) + 4(1-k)) \\ &= 0\end{aligned}$$

and

$$\begin{aligned} N(f) &= \frac{1}{4} \sum_{x \in \mathbb{Z}_4} |\det(I_5 - T_*(x)\mathfrak{D}_*)| \\ &= \frac{1}{4}(0 + 4|1-k| + 8|-1+k| + 4|1-k|) \end{aligned}$$

If we for instance assume that $k \geq 5$ then $N(f) = 4(k-1)$. So indeed, $|L(f)| \leq N(f)$ but $N(f)$ can be as large as we want.

Part II

The results

As is explained and motivated in the first part, in the following chapters we examine the Anosov relation for infra-nilmanifolds. Our main purpose is trying to generalize the Anosov theorem and in the case that this turns out to be not possible we provide the necessary counterexamples.

There are two possible ways of trying to generalize the Anosov theorem. Firstly, one can look for classes of manifolds, other than nilmanifolds, for which the relation holds for all continuous maps of the given manifold. This was, for instance, established by E. Keppelmann and C. McCord for exponential solvmanifolds ([31]). We try to do this for classes of infra-nilmanifolds. In Chapter 3 we examine for example infra-nilmanifolds with odd order holonomy group, in Chapter 5 infra-nilmanifolds with cyclic holonomy group and in Chapter 6 flat generalized Hantzsche-Wendt manifolds.

Secondly, one can search for classes of maps on a certain (type of) manifold for which the relation holds. For instance, S. Kwasik and K.B. Lee proved in [34] that the Anosov theorem holds for homotopic periodic maps of infra-nilmanifolds and C. McCord extended this to homotopic periodic maps of infra-solvmanifolds ([45]). Note that f is a homotopic periodic map if some power of f is homotopic to the identity and infra-solvmanifolds are manifolds which have a finite regular cover by a solvmanifold. A final example are the virtual unipotent maps introduced and studied in [42] by W. Malfait. In Chapter 3 we examine the Anosov relation for expanding maps and the new class of nowhere expanding maps which form an extension of the class of virtual unipotent maps. In the fourth chapter we examine the Anosov diffeomorphisms.

Chapter 3

Periodic sequences and infra-nilmanifolds with an odd order holonomy group

To prove the results in this chapter[2] we introduce the concept of a periodic sequence associated to a map. This sequence is used in an analogue way in three different situations. Firstly, we show that the Anosov theorem can be generalized to the much larger class of infra-nilmanifolds with odd order holonomy group. Secondly we look at two classes of maps: the well-known expanding maps and the by us introduced nowhere expanding maps. This last class is the complete opposite of the expanding maps, as will be clear from the definition. For these nowhere expanding maps we can easily show that the Anosov relation holds. Finally we establish a necessary and sufficient condition for expanding maps f of infra-nilmanifolds M to satisfy the Anosov relation. Namely, $N(f) = |L(f)|$ if and only if M is orientable.

This last result is very interesting since we can use it to decide about the (non)-validity of the Anosov theorem for non-orientable infra-nilmanifolds. However, not every (non-orientable) infra-nilmanifold admits an expanding map. In fact, in the last section we present an example of a non-orientable infra-nilmanifold, which does not admit expanding maps and for which the Anosov theorem holds for this specific manifold.

3.1 The Anosov theorem for infra-nilmanifolds with odd order holonomy group

In this section we show that the Anosov theorem holds for all infra-nilmanifolds M with odd order holonomy group F. To be able to prove

[2] The results of this chapter can also be found in [13].

this statement, we will associate to any element of F a certain sequence of elements of F. The construction of this sequence depends on the map f for which we want to prove that $N(f) = |L(f)|$.

Suppose that $f : M \to M$ is a continuous map, that $(\delta, \mathfrak{D})$ is a homotopy lift of f and that $T : F \to \mathrm{Aut}(G)$ is the associated holonomy representation. Now we exploit the fact that $(\delta, \mathfrak{D})$ is obtained from Theorem 1.10, so we can apply Corollary 1.13.

Using this we can construct for any $x_1 \in F$ a sequence of elements of F, say $x_1, x_2, \ldots$, such that for any $i \in \mathbb{N}_0$: $\mathfrak{D}_* T_*(x_i) = T_*(x_{i+1})\mathfrak{D}_*$. Since F is finite we know that we can construct this sequence in such a way that from a certain point onwards the sequence of elements becomes periodic. I.e. since F is finite we know that there has to exist a $j \geq 1$ and a $k \geq 1$ such that $x_{j+k} = x_j$. From that point we can take $x_{j+k+1} = x_{j+1}$, $x_{j+k+2} = x_{j+2}$, $\ldots$ (in general $x_{j+nk+l} = x_{j+l}$, for all $n, l \geq 0$) and still have the property that $\mathfrak{D}_* T_*(x_i) = T_*(x_{i+1})\mathfrak{D}_*$, $\forall i \geq 1$. So, the sequence we construct for a given x_1 is periodic, with period k, from a certain position j onwards.

Note that we do not claim that this sequence is unique, but it is always possible to construct such a sequence as described above. Let us illustrate the set-up of such a periodic sequence in an example.

Example 3.1. *Let E be the Bieberbach group generated by*

$$(e_1, I_4), \cdots (e_4, I_4), (a, A) = (\begin{pmatrix} 1/4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix})$$

where e_i stands for a column vector of length 4 with 1 in the i-th place and 0 everywhere else ($1 \leq i \leq 4$). It can be verified that E is indeed a Bieberbach group which fits into the following short exact sequence

$$1 \to \mathbb{Z}^4 \to E \to \mathbb{Z}_4 \to 1.$$

So we obtain a 4-dimensional flat manifold $M = E \backslash \mathbb{R}^4$ with holonomy group $\mathbb{Z}_4$. Then take

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{4} & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix})$$

which induces a map $f : M \to M$ (this is verified as before).
For this $(\delta, \mathfrak{D})$ we have that

$$\begin{aligned}(e_1 - 4e_2, I_4)(\delta, \mathfrak{D}) &= (\delta, \mathfrak{D})(e_1, I_4);\\ (a, A)(\delta, \mathfrak{D}) &= (\delta, \mathfrak{D})(e_2, I_4);\\ (-e_2 + a, A)(\delta, \mathfrak{D}) &= (\delta, \mathfrak{D})(a, A)\end{aligned}$$

Suppose that x generates the holonomy group of M, then we have that the periodic sequence associated to x is $x, x, x, \ldots$. On the other hand the periodic sequence for e is not unique: it can be taken equal to $e, e, e, \ldots$ but since $(a, A)(\delta, \mathfrak{D}) = (\delta, \mathfrak{D})(e_2, I_4)$ also equal to $e, x, x, \ldots$. As will be clear of the use of this periodic sequence, this non-uniqueness will not be a problem.

We will refer to the sequence $x_1, \ldots, x_j, \ldots, x_{j+k-1}, x_{j+k} = x_j, \ldots$ constructed as explained above as **a periodic sequence for x_1, associated to f, with period k starting from position j**. Using these sequences, we can now prove the following lemma.

Lemma 3.2. *Suppose M is an infra-nilmanifold with holonomy group F and associated holonomy representation $T : F \to \mathrm{Aut}(G)$. Suppose $f : M \to M$ is a continuous map and $(\delta, \mathfrak{D})$ is a homotopy lift of f. Let $x_1 \in F$ and let $x_1, x_2, x_3, \ldots$ be a periodic sequence for x_1 associated to f, with period k starting from position j. Then*

1. *$\forall i \in \mathbb{N}_0$: $\det(I_n - T_*(x_1)\mathfrak{D}_*) = \det(I_n - T_*(x_i)\mathfrak{D}_*)$;*
2. *$\mathfrak{D}_*^k T_*(x_j) = T_*(x_j)\mathfrak{D}_*^k$;*
3. *$\exists l \in \mathbb{N}_0$: $(T_*(x_j)\mathfrak{D}_*)^l = \mathfrak{D}_*^l$.*

Proof: We first prove that for any $i \geq 1$ we have that $\det(I_n - T_*(x_i)\mathfrak{D}_*) = \det(I_n - T_*(x_{i+1})\mathfrak{D}_*)$. Indeed,

$$\begin{aligned}\det(I_n - T_*(x_i)\mathfrak{D}_*) &= \det(T_*(x_i))\det((T_*(x_i))^{-1} - \mathfrak{D}_*)\\ &= \det(I_n - \mathfrak{D}_* T_*(x_i))\\ &= \det(I_n - T_*(x_{i+1})\mathfrak{D}_*)\end{aligned}$$

So for any element x_i of this periodic sequence we have that $\det(I_n - T_*(x_1)\mathfrak{D}_*) = \det(I_n - T_*(x_i)\mathfrak{D}_*)$.

To prove the second statement of this lemma, we compute

$$\begin{aligned}
\mathfrak{D}_*^k T_*(x_j) &= \mathfrak{D}_*^{k-1} T_*(x_{j+1})\mathfrak{D}_* \\
&= \mathfrak{D}_*^{k-2} T_*(x_{j+2})\mathfrak{D}_*^2 \\
&= \dots \\
&= \mathfrak{D}_* T_*(x_{j+k-1})\mathfrak{D}_*^{k-1} \\
&= T_*(x_{j+k})\mathfrak{D}_*^k \\
&= T_*(x_j)\mathfrak{D}_*^k.
\end{aligned}$$

To obtain the third claim, let us first look at the k-th power of $T_*(x_j)\mathfrak{D}_*$.

$$\begin{aligned}
(T_*(x_j)\mathfrak{D}_*)^k &= T_*(x_j)\mathfrak{D}_* T_*(x_j)\mathfrak{D}_*(T_*(x_j)\mathfrak{D}_*)^{k-2} \\
&= T_*(x_j)T_*(x_{j+1})\mathfrak{D}_*^2 T_*(x_j)\mathfrak{D}_*(T_*(x_j)\mathfrak{D}_*)^{k-3} \\
&= T_*(x_j)T_*(x_{j+1})T_*(x_{j+2})\mathfrak{D}_*^3(T_*(x_j)\mathfrak{D}_*)^{k-3} \\
&= \dots \\
&= T_*(x_j)T_*(x_{j+1})\cdots T_*(x_{j+k-1})\mathfrak{D}_*^k
\end{aligned}$$

Let $y = x_j x_{j+1}\cdots x_{j+k-1} \in F$. Then y is of finite order, say p, and analogously as before one can verify that $T_*(y)$ commutes with $\mathfrak{D}_*^k$. Therefore if $l = pk$, we obtain that $(T_*(x_j)\mathfrak{D}_*)^l = \mathfrak{D}_*^l$. □

With this lemma we can then prove the following theorem.

Theorem 3.3. *If M is an infra-nilmanifold with odd order holonomy group F, then $N(f) = |L(f)|$ for any continuous map $f : M \to M$.*

Proof: Suppose $f : M \to M$ is a continuous map of M and $(\delta, \mathfrak{D})$ is a homotopy lift of f. To use Theorem 2.5 we have to determine the sign of $\det(I_n - T_*(x_1)\mathfrak{D}_*)$ for any $x_1 \in F$. Construct a periodic sequence $x_1, \dots, x_{j+k-1}, x_{j+k} = x_j, \dots$ with period k starting from position j as explained above. Because of Lemma 3.2 it suffices to determine the sign of $\det(I_n - T_*(x_j)\mathfrak{D}_*)$ and we also know that $T_*(x_j)\mathfrak{D}_*^k = \mathfrak{D}_*^k T_*(x_j)$. We can pick a $P \in \mathrm{Gl}(n, \mathbb{R})$ which separates the eigenvalues of $\mathfrak{D}_*$:

$$P\mathfrak{D}_*P^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

with $D_1 \in M_m(\mathbb{R})$ (resp. $D_2 \in M_{n-m}(\mathbb{R})$) and for any eigenvalue λ of D_1 (resp. any eigenvalue λ of D_2) we have that $|\lambda| \leq 1$ (resp. $|\lambda| > 1$). The construction of D_1 and D_2 implies that D_1^k and D_2^k must have distinct eigenvalues and therefore we have that $T_*(x_j)\mathfrak{D}_*^k = \mathfrak{D}_*^k T_*(x_j)$ implies

$$PT_*(x_j)P^{-1} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

with $T_1 \in M_m(\mathbb{R})$ and $T_2 \in M_{n-m}(\mathbb{R})$. So for any $x_1 \in F$:

$$\begin{aligned} \det(I_n - T_*(x_1)\mathfrak{D}_*) &= \det(I_n - T_*(x_j)\mathfrak{D}_*) \\ &= \det(I_m - T_1 D_1)\det(I_{n-m} - T_2 D_2) \end{aligned} \tag{3.1}$$

Lemma 3.2 guarantees the existence of a $l > 0$ such that $(T_1D_1)^l = D_1^l$. This implies that for any eigenvalue λ of T_1D_1, we also have that $|\lambda| \leq 1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the set of eigenvalues of T_1D_1 (with each eigenvalue listed as many times as its algebraic multiplicity), then $|\lambda_i| \leq 1$ $(\forall i)$ implies that

$$\det(I_m - T_1D_1) = (1-\lambda_1)\cdots(1-\lambda_m) \geq 0$$

So the first factor in (3.1) does not play a role in determining the sign of $\det(I_n - T_*(x_1)\mathfrak{D}_*)$.

Note that for any non-real eigenvalue λ_i, we know that $\overline{\lambda_i}$ is also an eigenvalue which implies that $(1-\lambda_i)(1-\overline{\lambda_i}) \geq 0$.

One can easily verify that the sign of the second factor depends on the number of real, positive eigenvalues of T_2D_2 (note that Lemma 3.2 now implies that any eigenvalue λ of T_2D_2 is of modulus > 1). Let us therefore denote the number (counted with multiplicity) of real, positive eigenvalues of D_2 by p and those of T_2D_2 by q. If we can now show that $q \equiv p \bmod 2$, then all determinants must have the same sign and this would finish the proof.
Since F is of odd order, we have that $T_*(x_j)$ is of odd, finite order. Hence, also T_2 is of odd order which implies that $\det(T_2) = 1$. Therefore we have that

$$\det(T_2D_2) = \det(T_2)\det(D_2) = \det(D_2). \tag{3.2}$$

Now, $\det(D_2) \neq 0$ since for any eigenvalue λ of D_2 yields that $|\lambda| > 1$. So the equality above a fortiori implies that both determinants have the same sign. So, modulo 2, the matrices D_2 and T_2D_2 must have the same number of real, negative eigenvalues. But, as the number of non-real eigenvalues is even for both matrices, this also implies that $q \equiv p \bmod 2$. □

Remark 3.4. *The condition that F is of odd order is crucial since for instance Anosov constructed in [1] a counter example on the Klein bottle (which is a flat manifold having $\mathbb{Z}_2$ as holonomy group). See also Chapter 5 for a detailed study of infra-nilmanifolds with cyclic holonomy groups or Part 3 for more counterexamples.*

3.2 Classes of maps for which the Anosov theorem holds

In the proof of Theorem 3.3 we separated the eigenvalues of the differential $\mathfrak{D}_*$ of the linear part of the homotopy lift. Because of this separation, we were able to construct matrices T_1 and T_2 (using the notations from the proof of Theorem 3.3). Now, in general, it is not possible to use the same techniques for infra-nilmanifolds with a holonomy group of even order. Since in that case -1 can be an eigenvalue of T_2 and so $\det(T_2)$ could be equal to -1. Moreover, because of the separation of $\mathfrak{D}_*$, we do not longer have control on how many times -1 appears as an eigenvalue of T_2.

In this section we avoid the separation of $\mathfrak{D}_*$ by looking at special classes of maps. Namely we assume that either D_1 or D_2 does not appear.

3.2.1 The Anosov relation for expanding maps

The first class of maps that we want to study are the expanding maps. Let us therefore first recall the definition.

Definition 3.5. *A C^1-self-map $f : M \to M$ on a closed smooth manifold M is said to be an expanding map if there exist constants $C > 0$ and $\mu > 1$ such that*

$$\|Df^n(v)\| \geq C\mu^n\|v\|, \forall v \in TM$$

for some Riemannian metric $\| \ \|$ on M.

It follows from the work of M. Gromov ([25]) that any expanding map of a compact manifold is topologically conjugated to an expanding infra-nilmanifold endomorphism. This means that for the homotopy lift $(\delta, \mathfrak{D})$ of an expanding map on an infra-nilmanifold we have that the modulus of all eigenvalues of $\mathfrak{D}_*$ is larger than 1.

The following theorem is based on the orientability of the infra-nilmanifold M which is completely determined by Proposition 1.6. Using this proposition we can now prove the following result about expanding maps of infra-nilmanifolds.

Theorem 3.6. *Suppose $f : M \to M$ is an expanding map of an infra-nilmanifold M. Then $N(f) = |L(f)|$ if and only if M is orientable.*

Proof: Suppose $f : M \to M$ is an expanding map and suppose $(\delta, \mathfrak{D})$ is a homotopy lift of f. We will use Theorem 2.5 and therefore we have to determine the sign of $\det(I_n - T_*(x_1)\mathfrak{D}_*)$ for any $x_1 \in F$. Construct a periodic sequence $x_1, \ldots, x_{j+k-1}, x_{j+k} = x_j, \ldots$ as before. Because of Lemma 3.2 is suffices to look at $\det(I_n - T_*(x_j)\mathfrak{D}_*)$ and we also know that $T_*(x_j)\mathfrak{D}_*^k = \mathfrak{D}_*^k T_*(x_j)$.

Let us first assume that M is orientable and so for any $x \in F$ we have that $\det(T_*(x)) = 1$. Therefore

$$\det(T_*(x_j)\mathfrak{D}_*) = \det(T_*(x_j))\det(\mathfrak{D}_*) = \det(\mathfrak{D}_*).$$

Recall that $\det(\mathfrak{D}_*) \neq 0$ since $|\lambda| > 1$ for any eigenvalue λ of $\mathfrak{D}_*$. Completely analogous as in the proof of Theorem 3.3 one can now verify that this implies that all the determinants have the same sign.

Let us now assume that M is non-orientable and so there exists an $x_1 \in F$ such that $\det(T_*(x_1)) = -1$. Then for any x_i of a periodic sequence $x_1, \ldots, x_{j+k-1}, x_{j+k} = x_j, \ldots$ we have that $\det(T_*(x_i)) = -1$. Indeed, by construction we have for any i that

$$\begin{aligned}\det(\mathfrak{D}_*)\det(T_*(x_i)) &= \det(\mathfrak{D}_* T_*(x_i))\\ &= \det(T_*(x_{i+1})\mathfrak{D}_*)\\ &= \det(T_*(x_{i+1}))\det(\mathfrak{D}_*).\end{aligned}$$

Since $\det(\mathfrak{D}_*) \neq 0$, this implies that $\det(T_*(x_i)) = \det(T_*(x_{i+1}))$ for any i.
So we can assume that $\det(T_*(x_j)) = -1$ and

$$\det(T_*(x_j)\mathfrak{D}_*) = \det(T_*(x_j))\det(\mathfrak{D}_*) = -\det(\mathfrak{D}_*).$$

Denote the number of positive, real eigenvalues of $\mathfrak{D}_*$ by p and of $T_*(x_j)\mathfrak{D}_*$ by q. Then one can, again analogously as in the proof of Theorem 3.3, easily verify that $q \not\equiv p \bmod 2$. This implies that $\det(I_n - \mathfrak{D}_*)$

and $\det(I_n - T_*(x_j)\mathfrak{D}_*)$ have a different sign. Note that these determinants can not be equal to zero, since for any eigenvalue λ of $\mathfrak{D}_* : |\lambda| > 1$ and because of the last statement of Lemma 3.2 the same holds for $T_*(x_j)\mathfrak{D}_*$. Theorem 2.5 then finishes the proof. □

So this theorem implies that the Anosov relation for expanding maps is completely determined by the orientability of the infra-nilmanifold. It also implies that if M is a non-orientable manifold which admits an expanding map f, then the Anosov theorem no longer holds for M.

This is an interesting result since in [22] it was shown that all flat manifolds admit an expanding map. Moreover the same is true for 2-step infra-nilmanifolds ([36]), but on the other hand there are c-step infra-nilmanifolds (for $c > 2$) which do not admit expanding maps. See for instance [20] for a 3-step infra-nilmanifold which does not admit expanding maps. Since the above result will be a very useful result later on, we state it formally.

Proposition 3.7. *If M is a non-orientable flat manifold or a non-orientable 2-step infra-nilmanifold then the Anosov theorem does not hold for M.*

In general however this is no longer true and to prove this we construct a non-orientable infra-nilmanifold for which the Anosov theorem holds (and so a fortiori this manifold does not admit expanding maps). This example will be also useful in the following chapters and will show on more then one occasion that infra-nilmanifolds are much more complicated than flat manifolds. The example is presented in the last section of this chapter.

3.2.2 The Anosov relation for nowhere expanding maps

In this section we investigate the complete opposite of the class of expanding maps and introduce what we call the nowhere expanding maps.

Definition 3.8. *Let $f : M \to M$ be a continuous map of an infra-nilmanifold M, with a homotopy lift $(\delta, \mathfrak{D})$. Then f is said to be a nowhere expanding map, if $|\lambda| \leq 1$ for all eigenvalues λ of $\mathfrak{D}_*$.*

Remark 3.9. *The class of nowhere expanding maps also contains the virtually unipotent maps (for which it is requested that all eigenvalues λ of $\mathfrak{D}_*$ are of modulus 1) introduced in [42].*

The class of nowhere expanding maps is in fact very closely related to the class of virtually unipotent maps. Indeed, one might be tempted to think that there are many possibilities for the moduli of the eigenvalues of $\mathfrak{D}_*$ since we only request them to be less than or equal to 1. However, the following result shows that in the case of nowhere expanding maps these moduli are either 1 or 0 and nothing in between is possible.

Lemma 3.10. *Let $f : M \to M$ be a nowhere expanding map on an infra-nilmanifold M, with homotopy lift $(\delta, \mathfrak{D})$. Let λ be an eigenvalue of $\mathfrak{D}_*$, then either $|\lambda| = 1$ or $\lambda = 0$.*

Proof: Let $f_* : E \to E$ be the homomorphism induced by f on the fundamental group E of $M = E\backslash G$. The group E is an almost-Bieberbach group, and we consider it as a subgroup of $G \rtimes \mathrm{Aut}(G)$. Now, let $\Lambda \subseteq G \cap E$ be a subgroup of finite index in E, satisfying $f_*(\Lambda) \subseteq \Lambda$. It follows from Theorem 1.10, that

$$\forall \lambda \in \Lambda : \ f_*(\lambda) = \delta \mathfrak{D}(\lambda) \delta^{-1} = (\mu(\delta) \circ \mathfrak{D})(\lambda)$$

where $\mu(\delta)$ denotes conjugation with δ.
We will now choose a special basis of the Lie algebra $\mathfrak{g}$ of G, by determining a specific generating set for the lattice Λ of G. Let $\gamma_1(\Lambda) = \Lambda$ and $\gamma_{i+1}(\Lambda) = [\Lambda, \gamma_i(\Lambda)]$. As Λ is a nilpotent group, $\gamma_{c+1}(\Lambda) = 1$ for some c. For any $i \geq 1$, we take

$$\Lambda_i = \sqrt{\gamma_i(\Lambda)} = \{x \in \Lambda \mid x^k \in \gamma_i(\Lambda), \text{ for some } k > 0\}.$$

Then Λ_i is a fully invariant subgroup of Λ, containing $\gamma_i(\Lambda)$ as a subgroup of finite index and we have that Λ_i/Λ_{i+1} is free abelian of finite rank, say k_i.
We now choose a set of generators

$$a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{2,k_2}, a_{3,1}, \ldots, a_{c,k_c}$$

for Λ, in such a way that $a_{i,j} \in \Lambda_i$, and the natural projections of $a_{i,1}, a_{i,2}, \ldots, a_{i,k_i}$ freely generate $\Lambda_i/\Lambda_{i+1} \cong \mathbb{Z}^{k_i}$.
As G is a nilpotent Lie group, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism and we denote its inverse by log. The set $A_{1,1} = \log(a_{1,1}), \ldots, A_{c,k_c} = \log(a_{c,k_c})$ forms a basis of $\mathfrak{g}$. With respect to this basis, $(\mu(\delta) \circ \mathfrak{D})_*$ has a matrix representation of the form

$$\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ * & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & B_c \end{pmatrix}, \tag{3.3}$$

where B_i is a $(k_i \times k_i)$-matrix. In fact, B_i represents the morphism, induced by f_* on the free abelian group Λ_i/Λ_{i+1} and hence B_i is a matrix with integral entries. Now, as $\mathfrak{D}$ only differs from $\mu(\delta) \circ \mathfrak{D}$ by an inner conjugation of G, the matrix representing $\mathfrak{D}_*$ is also of the form (3.3) with the same integral blocks B_i on the diagonal (but with different entries below the diagonal). It follows that the eigenvalues of $\mathfrak{D}_*$ are exactly the same as the eigenvalues of the integral matrix

$$\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_c \end{pmatrix}.$$

The characteristic polynomial $p(x)$ of this matrix (and hence also of the matrix representing $\mathfrak{D}_*$) is of the form

$$p(x) = x^k(a_l x^l + \cdots + a_1 x + a_0)$$

with $a_i \in \mathbb{Z}$, $a_0 \neq 0$ and k is the multiplicity of the eigenvalue 0. This means that the non-zero eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_l$ satisfy $\lambda_1\lambda_2\cdots\lambda_l = a_0$. As it is given that $|\lambda_i| \leq 1$ $(1 \leq i \leq l)$ and $|a_0| \geq 1$ (a_0 is a non-zero integer), we must have that $|\lambda_i| = 1$ $(1 \leq i \leq l)$ (and $a_0 = \pm 1$). □

We are now ready to prove that the Anosov relation holds for any nowhere expanding map. This results generalizes Theorem 4.3 of [42], where the analogous result for virtually unipotent maps was obtained.

Theorem 3.11. *Let $f : M \to M$ be a nowhere expanding map on an infra-nilmanifold M, then $N(f) = L(f)$.*

Proof: Let $(\delta, \mathfrak{D})$ be a homotopy lift of f. We will use Theorem 2.5 and therefore we have to determine the sign of $\det(I_n - T_*(x_1)\mathfrak{D}_*)$ for any $x_1 \in F$. Construct a periodic sequence $x_1, \ldots, x_{j+k-1}, x_{j+k} = x_j, \ldots$ with period k and starting point j as explained in the previous section and suppose $\lambda_1, \ldots \lambda_n$ are the eigenvalues of $T_*(x_j)\mathfrak{D}_*$ (with again each eigenvalue listed as many times as its algebraic multiplicity). Because of Lemma 3.2 it suffices to look at $\det(I_n - T_*(x_j)\mathfrak{D}_*)$. Moreover the last statement of Lemma 3.2 implies that for any i, $1 \leq i \leq n$, we have that $|\lambda_i| \leq 1$ since for any eigenvalue λ of $\mathfrak{D}_* : |\lambda| \leq 1$. So for any $x_1 \in F$:

$$\det(I_n - T_*(x_1)\mathfrak{D}_*) = \det(I_n - T_*(x_j)\mathfrak{D}_*) = (1-\lambda_1)\cdots(1-\lambda_n) \geq 0.$$

Theorem 2.5 then finishes the proof. □

Example 3.12. *The map f introduced in Example 3.1 is an example of an nowhere expanding map since the eigenvalues of $\mathfrak{D}$ are* 0 *and* $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ *and indeed* $\det(I_4 - A^k\mathfrak{D}) = 1 \geq 0$ *for any k.*

3.3 Infra-nilmanifolds are more complicated

In this section we construct a 7-dimensional non-orientable infra-nilmanifold M_1 with holonomy group $\mathbb{Z}_2$ for which the Anosov theorem holds (and hence M_1 does not admit any expanding map, see Proposition 3.7). This M_1 implies that the results obtained in the previous section are not true in general for k-step infra-nilmanifolds with $k > 2$. Let $\mathfrak{g}$ be the 7-dimensional Lie algebra with basis $X_1, X_2, \ldots, X_7$ where the non-zero Lie brackets between basis vectors are given by

$$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_1, X_4] = X_5,\ [X_1, X_5] = X_6$$

$$[X_2, X_3] = -X_7,\ [X_2, X_7] = -X_5 - X_6,\ [X_3, X_7] = -X_6$$

Remark 3.13. *The reason for choosing this Lie algebra lies in the fact that we want to base our example on a Lie group with as few as possible automorphisms, certainly without any expanding automorphisms. The above Lie algebra is a characteristically nilpotent Lie algebra and hence any automorphism of this Lie algebra has only eigenvalues of modulus 1. Dimension 7 is the lowest possible dimension in which characteristically nilpotent Lie algebras occur. See [24, Chapter 2, section III] for more details.*
Note that we do not claim that there does not exist analogue examples in lower dimensions, since we did not examine this.

There is a faithful matrix representation of this Lie algebra, which is given by

$$\rho : \mathfrak{g} \to M_8(\mathbb{R}^n) : x_1X_1 + x_2X_2 + \cdots + x_7X_7 \mapsto$$

$$\begin{pmatrix} 0 & x_1 & -\frac{2}{3}(x_2+x_3) & 0 & \frac{x_7}{3} & \frac{x_7}{3} & 0 & x_6 \\ 0 & 0 & -\frac{2}{3}x_2 & x_1 & 0 & \frac{x_7}{3} & 0 & x_5 \\ 0 & 0 & 0 & 0 & -\frac{x_2}{2} & \frac{x_3}{2} & 0 & x_7 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

One might use this matrix representation in checking the claims which follow.
Let $\exp : \mathfrak{g} \to G$ denote the exponential map from the nilpotent Lie algebra $\mathfrak{g}$ to the corresponding simply connected, connected nilpotent Lie group G. (Note that ρ lifts to a matrix representation of G). Consider

$$a_1 = \exp(X_1),\ a_2 = \exp(X_2),\ a_3 = \exp(\frac{1}{2}X_3),\ a_4 = \exp(\frac{1}{8}X_4),$$

$$a_5 = \exp(\frac{1}{48}X5),\ a_6 = \exp(\frac{1}{384}X_6) \text{ and } a_7 = \exp(\frac{1}{4}X_7).$$

Let $\mathfrak{T}$ be the automorphism of G, whose differential $\mathfrak{T}_* : \mathfrak{g} \to \mathfrak{g}$ satisfies:

$$\mathfrak{T}_*(X_1) = X_1,\ \mathfrak{T}_*(X_2) = -X_2,\ \mathfrak{T}_*(X_3) = -X_3,\ \mathfrak{T}_*(X_4) = -X_4,$$

$$\mathfrak{T}_*(X_5) = -X_5,\ \mathfrak{T}_*(X_6) = -X_6 \text{ and } \mathfrak{T}_*(X_7) = X_7.$$

Let $\alpha \in \operatorname{Aff}(G)$ be the element $(\exp(\frac{1}{2}X_1), \mathfrak{T})$. The subgroup E of $\operatorname{Aff}(G)$ generated by $a_1, \ldots, a_7$ and α has a presentation of the form

$$E = \left\langle a_1, a_2, \ldots, a_7, \alpha \;\middle|\; \begin{array}{ll}
[a_1,a_2] = a_3^2 a_4^{-4} a_5^{16} a_6^{112} a_7^2 & [a_3,a_4] = 1 \\
[a_1,a_3] = a_4^4 a_5^{-12} a_6^{32} & [a_3,a_5] = 1 \\
[a_1,a_4] = a_5^6 a_6^{-24} & [a_3,a_6] = 1 \\
[a_1,a_5] = a_6^8 & [a_3,a_7] = a_6^{-48} \\
[a_1,a_6] = 1 & [a_4,a_5] = 1 \\
[a_1,a_7] = 1 & [a_4,a_6] = 1 \\
[a_2,a_3] = a_5^{-12} a_6^{-144} a_7^{-2} & [a_4,a_7] = 1 \\
[a_2,a_4] = 1 & [a_5,a_6] = 1 \\
[a_2,a_5] = 1 & [a_5,a_7] = 1 \\
[a_2,a_6] = 1 & [a_6,a_7] = 1 \\
[a_2,a_7] = a_5^{-12} a_6^{-96} & \\
\alpha^2 = a_1 & \alpha a_1 = a_1 \alpha \\
\alpha a_2 = a_2^{-1} \alpha a_3 a_4^{-1} a_5^5 a_6^{47} a_7 & \alpha a_3 = a_3^{-1} \alpha a_4^2 a_5^{-3} a_6^4 \\
\alpha a_4 = a_4^{-1} \alpha a_5^3 a_6^{-6} & \alpha a_5 = a_5^{-1} \alpha a_6^4 \\
\alpha a_6 = a_6^{-1} \alpha & \alpha a_7 = a_7 \alpha
\end{array} \right\rangle.$$

This group E is an almost-crystallographic subgroup of $\operatorname{Aff}(G)$. Moreover, E is torsion-free and hence it is the fundamental group of an infra-nilmanifold M_1. The holonomy group is $\mathbb{Z}_2$ and the infra-nilmanifold is non-orientable ($\det(\mathfrak{T}_*) = -1$, see Proposition 1.6).
Consider now any self-map f of M_1, inducing an endomorphism θ on the fundamental group $\pi_1(M_1) = E$ and with homotopy lift $(\delta, \mathfrak{D})$. Then,

we can distinguish two possibilities for $\mathfrak{D}_*$, namely $\mathfrak{D}_*\mathfrak{T}_* = \mathfrak{T}_*\mathfrak{D}_*$ or $\mathfrak{D}_*\mathfrak{T}_* = \mathfrak{D}_*$.

Any endomorphism $\mathfrak{D}_*$ of $\mathfrak{g}$ is completely determined by the images of its generators X_1 and X_2. Let us use the following notation

$$\mathfrak{D}_*(X_1) = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_7 X_7$$

and

$$\mathfrak{D}_*(X_2) = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_7 X_7$$

Case 1: $\mathfrak{D}_*\mathfrak{T}_* = \mathfrak{T}_*\mathfrak{D}_*$ This case forces all of the parameters $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_1, \beta_7$ to be zero. Moreover, this case happens exactly when $\theta(\alpha) = a_1^{k_1} a_2^{k_2} \ldots a_7^{k_7}\alpha$ for some integers $k_i \in \mathbb{Z}$. It then follows that $\theta(a_1) = \theta(\alpha)^2 = a_1^{2k_1+1} a_2^{l_2} \cdots a_7^{l_7}$ for some other integers $l_i \in \mathbb{Z}$. This implies that $\alpha_1 = 2k_1 + 1 \neq 0$. It follows that in this case $\mathfrak{D}_*$ satisfies:

$$\mathfrak{D}_*(X_1) = (2k_1 + 1)X_1 + \alpha_7 X_7$$

and

$$\mathfrak{D}_*(X_2) = \beta_2 X_2 + \beta_3 X_3 + \cdots + \beta_6 X_6.$$

If we now require that $\mathfrak{D}_*$ actually determines an endomorphism of $\mathfrak{g}$, then there are 3 subcases to consider:

- $\beta_2 = \beta_3 = 0$ in which case $\mathfrak{D}_*$ has a matrix representation, w.r.t. the basis $X_1, X_2, \ldots, X_7$ of the form

$$\begin{pmatrix} 2k_1+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \end{pmatrix}$$

  It is obvious that $\det(I_7 - \mathfrak{D}_*) = \det(I_7 - \mathfrak{T}_*\mathfrak{D}_*) = -2k_1$ in this case.
- Another possibility is that $\beta_2 = \alpha_1 = 1$. Now $\mathfrak{D}_*$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & 1 & 0 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 \\ * & * & * & * & * & 1 & 0 \\ * & * & * & * & * & * & 1 \end{pmatrix}$$

which again implies that $\det(I_7 - \mathfrak{D}_*) = \det(I_7 - \mathfrak{T}_*\mathfrak{D}_*) = 0$.

- Finally it is possible that $\alpha_1 = 1$ and $\beta_2 = -1$. Now, $\mathfrak{D}_*$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -1 & 0 & 0 & 0 & 0 & 0 \\ * & * & -1 & 0 & 0 & 0 & 0 \\ * & * & * & -1 & 0 & 0 & 0 \\ * & * & * & * & -1 & 0 & 0 \\ * & * & * & * & * & -1 & 0 \\ * & * & * & * & * & * & 1 \end{pmatrix}$$

which again implies that $\det(I_7 - \mathfrak{D}_*) = \det(I_7 - \mathfrak{T}_*\mathfrak{D}_*) = 0$.

Case 2: $\mathfrak{D}_*\mathfrak{T}_* = \mathfrak{D}_*$ This case forces $\beta_1 = \beta_2 = \cdots = \beta_7 = 0$. Hence $\mathfrak{D}_*$ is of the form

$$\begin{pmatrix} k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_7 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Again we obtain that $\det(I_7 - \mathfrak{D}_*) = \det(I_7 - \mathfrak{T}_*\mathfrak{D}_*) = 1 - k_1$.

As before we can use Theorem 2.5 to conclude that the Anosov theorem holds on this infra-nilmanifold.

Note that this already shows in two ways that infra-nilmanifolds are much more complicated then flat manifolds. Firstly, there are no non-orientable flat manifolds for which the Anosov theorem holds.
Secondly, in Chapter 5 we will show that the Anosov theorem never holds for flat manifolds with $\mathbb{Z}_2$ as holonomy group. This example shows that this no longer holds in general for infra-nilmanifolds with $\mathbb{Z}_2$ as holonomy group.

Chapter 4

Anosov diffeomorphisms

In the previous chapter we examined two ('opposite') classes of maps which could be defined based on their eigenvalues of the homotopy lifts. In this chapter we examine another class of maps, namely the well studied class of Anosov diffeomorphisms. This class differs from what was studied in the previous chapter because, as we show later on, for these maps we always have both a D_1 and a D_2 (using the notations of the proof of Theorem 3.3). To prove this we start this chapter by recalling an algebraic characterization of Anosov diffeomorphisms.

A motivation for the examination of the Anosov relation for Anosov diffeomorphisms is the fact that these maps play an important role in dynamics. They could for instance be used to form nice examples of structurally stable dynamical systems, which also implies that $MF(f)$ is very interesting information. However it is important to note that not every compact manifold admits Anosov diffeomorphisms. Up till now the only known examples of Anosov diffeomorphisms are maps of infra-nilmanifolds. Moreover it has been conjectured that every Anosov diffeomorphism is topologically conjugated to an Anosov diffeomorphism of an infra-nilmanifold (see [43]) .

H. Porteous gave in [49] a complete characterization of the flat manifolds admitting Anosov diffeomorphisms. Unfortunately there is no such characterization for infra-nilmanifolds in general. Moreover J. Lauret argued in [35] that a reasonable classification of nilmanifolds admitting Anosov diffeomorphisms, and so a fortiori also of infra-nilmanifolds, is not possible. Therefore we focus on Anosov diffeomorphisms of n-dimensional flat manifolds.

In [41], it was proved that if M admits Anosov diffeomorphisms, then its first Betti number $b_1(M)$ satisfies one of the following: $b_1(M) = 0$, $2 \leq b_1(M) \leq n-2$ or $b_1(M) = n$ (and all situations occur). In the last case M is a torus and so $N(f) = |L(f)|$ for every continuous self-map f of M. For the other cases, we investigate the possibility of constructing a flat manifold M, with prescribed first Betti number, such that on the one hand M admits an Anosov diffeomorphism f with $N(f) \neq |L(f)|$ and on the other hand M also supports an Anosov diffeomorphism g satisfying $N(g) = |L(g)|$.
In the second section we show that for the case of non-primitive flat manifolds M, i.e. $b_1(M) \neq 0$, it turns out that this is always possible except when $b_1(M) = n-2$ (Theorem 4.6). In this latter case we show that $N(f) \neq |L(f)|$ for each flat manifold M (admitting an Anosov diffeomorphism) and each Anosov diffeomorphism f on M (Theorem 4.10). Primitive flat manifolds which admit Anosov diffeomorphisms only exist from dimension 6 onwards ([49]). In the third section we will construct such primitive flat manifolds and the desired Anosov diffeomorphisms f and g in any dimension $n > 6$ (Theorem 4.14). For each 6-dimensional primitive flat manifold M admitting an Anosov diffeomorphism however, we show that $N(f) = |L(f)|$ for all Anosov diffeomorphisms f of M (Theorem 4.16).

Finally we stress that we do not present an exhaustive examination of all possible Anosov diffeomorphisms of flat manifolds. Based on the obtained results, we obviously expect for $f : M \to M$ of flat manifolds that the property of f being an Anosov diffeomorphism has not much influence on f satisfying the Anosov relation. Except for some boundary cases for which we then present an exhaustive examination.[3]

4.1 Algebraic characterization

Let $f : M \to M$ be a diffeomorphism of a flat n-dimensional manifold M. We say that f is an affine diffeomorphism of M if the lifting of f to the universal cover $\mathbb{R}^n$ of M belongs to $\mathrm{Aff}(\mathbb{R}^n)$. In fact, such a lifting then automatically belongs to the normalizer $N_{\mathrm{Aff}(\mathbb{R}^n)}(E)$ of E. An affine diffeomorphism f of M is hyperbolic if and only if f lifts to an affine transformation $(a, A) \in \mathrm{Aff}(\mathbb{R}^n)$ with A hyperbolic (i.e. having no eigenvalues of absolute value 1).

[3] The results of this chapter can also be found in [16].

Definition 4.1. *A diffeomorphism $f : M \to M$ of a closed smooth manifold is an Anosov diffeomorphism if and only if tangent space TM decomposes as a direct sum of a contracting and an expanding part. That is, there is a continuous splitting $TM = E^s \oplus E^u$ such that for some (and hence any) any Riemannian metric on M, there are constants c and λ ($c > 0, 1 < \lambda$) such that for all $r \in \mathbb{N}_0$, $\|df^r(v)\| \geq c\lambda^r\|v\|$ (for all $v \in E^u$) and $\|df^r(w)\| \leq c^{-1}\lambda^{-r}\|w\|$ (for all $w \in E^s$).*

Such maps are studied in e.g. [23] and [43]. The examination of Anosov diffeomorphisms of flat manifolds can be converted into a pure algebraic problem. Namely, each hyperbolic diffeomorphism of a flat Riemannian manifold M defines an Anosov diffeomorphisms on M and conversely, we show in the following lemma that each Anosov diffeomorphism of M is homotopic to a hyperbolic diffeomorphism of M.

Lemma 4.2. *If $f : M \to M$ is an Anosov diffeomorphism of a flat manifold M, then f is homotopic to a hyperbolic diffeomorphism $g : M \to M$.*

Proof: In [43] Manning showed that each Anosov diffeomorphism f of M is topologically conjugated with a hyperbolic diffeomorphism g of M. Thus there exists a homeomorphism $h : M \to M$ such that $f = hgh^{-1}$. Since h is a homeomorphism of M, the induced map $h_* : \pi_1(M) \to \pi_1(M)$ is an isomorphism. Then because of the second theorem of Bieberbach there exists a $(\delta, \mathfrak{D}) \in \mathrm{Aff}(\mathbb{R}^n)$ such that h_* is exactly the conjugation with $(\delta, \mathfrak{D})$ restricted to $\pi_1(M)$. This $(\delta, \mathfrak{D})$ induces an affine diffeomorphism $\tilde{h}$ on M. Since M is aspherical, h and $\tilde{h}$ are homotopic ([54, page 225]). So f is homotopic to $\tilde{h}g\tilde{h}^{-1}$. Suppose g is induced by a $(\delta_1, \mathfrak{D}_1) \in \mathrm{Aff}(\mathbb{R}^n)$ with $\mathfrak{D}_1$ hyperbolic, then $\mathfrak{D}\mathfrak{D}_1\mathfrak{D}^{-1}$ is also hyperbolic and $\tilde{h}g\tilde{h}^{-1}$ is a hyperbolic diffeomorphism. □

A complete characterization of flat Riemannian manifolds supporting Anosov diffeomorphisms, due to H. Porteous ([49]), is the following:

Theorem 4.3. *A n-dimensional flat manifold M with holonomy group F and associated holonomy representation $T : F \to \mathrm{Gl}(n, \mathbb{Z})$ admits an Anosov diffeomorphism if and only if each $\mathbb{Q}$-irreducible component of T of multiplicity one is reducible over $\mathbb{R}$.*

Finally we present a criterion to calculate the first Betti number of a flat n-dimensional manifold M. Since we can see $\pi_1(M)$ as a subgroup of $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{Gl}(n, \mathbb{R})$, we may assume that $\pi_1(M)$ is generated by

$(a_1, A_1), \ldots, (a_k, A_k)$. The first Betti number $b_1(M)$ of M, or equivalently the first Betti number $b_1(E)$ of E, is by definition the rank of the first homology group $H_1(M, \mathbb{Z})$ (or equivalently of $H_1(E, \mathbb{Z})$). Because of [12, Remark 6.4.16] we know that

$$b_1(M) = n - rank(A_1 - I_n, \ldots, A_k - I_n),$$

4.2 Non-primitive flat manifolds

As mentioned already in the introduction, if a flat manifold M admits an Anosov diffeomorphism, its first Betti number $b_1(M)$ must satisfy $b_1(M) = 0,\ 2 \leq b_1(M) \leq n-2$ or $b_1(M) = n$. In this section we consider the case $2 \leq b_1(M) \leq n-2$ and we do this by first looking at $b_1(M) < n-2$ and then at $b_1(M) = n-2$.

4.2.1 Flat n-dimensional manifolds with first Betti number smaller than $n-2$

For any integer $n \geq 2$ and any integer k satisfying $1 \leq k \leq n-1$, we define the group $E_{n,k}$ generated by

$$(e_i, I_n) \text{ and } (a, A) = (\begin{pmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix})$$

with 1 in the i-th place and 0 everywhere else for e_i $(1 \leq i \leq n)$. One easily verifies that $E_{n,k}$ is torsion-free and $\mathbb{Z}^n$ is maximal abelian in $E_{n,k}$, because the associated holonomy representation clearly is faithful. Thus $E_{n,k}$ is a Bieberbach group fitting into the following short exact sequence:

$$1 \to \mathbb{Z}^n \to E_{n,k} \to \mathbb{Z}_2 \to 1.$$

$M_{n,k} = E_{n,k} \backslash \mathbb{R}^n$ is a n-dimensional flat manifold with first Betti number equal to k. The affine diffeomorphisms of $M_{n,k}$ are exactly those diffeomorphisms of $M_{n,k}$ lifting to a $(\delta, \mathfrak{D}) \in \mathrm{Aff}(\mathbb{R}^n)$ and normalizing $E_{n,k}$. In the next lemma we therefore characterize $N_{\mathrm{Aff}(\mathbb{R}^n)}(E_{n,k})$.

Lemma 4.4. *Suppose* $(\delta, \mathfrak{D}) \in \mathrm{Aff}(\mathbb{R}^n)$, *then*

$$(\delta, \mathfrak{D}) \in N_{\mathrm{Aff}(\mathbb{R}^n)}(E_{n,k}) \Leftrightarrow \delta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \mathfrak{D} = (d_{ij}) = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

with $x_1, \ldots, x_k \in \mathbb{R}$, $x_{k+1}, \ldots x_n \in \frac{1}{2}\mathbb{Z}$, $D_1 \in \mathrm{Gl}(k, \mathbb{Z})$, $D_2 \in \mathrm{Gl}(n - k, \mathbb{Z})$ *and* $d_{11} \in 1 + 2\mathbb{Z}$, $d_{21}, \ldots d_{k1} \in 2\mathbb{Z}$.

Proof: Suppose $\delta = (x_1, \ldots, x_n)^t$ with $x_i \in \mathbb{R}$ and $\mathfrak{D} = (d_{ij}) \in \mathrm{Gl}(n, \mathbb{R})$. First we consider the conjugates of (e_i, I_n):

$$\begin{aligned}(\delta, \mathfrak{D})(e_i, I_n)(\delta, \mathfrak{D})^{-1} &= (\delta + \mathfrak{D}e_i, \mathfrak{D})(-\mathfrak{D}^{-1}\delta, \mathfrak{D}^{-1}) \\ &= (\mathfrak{D}e_i, I_n).\end{aligned}$$

This belongs to $E_{n,k}$ if and only if all the elements of $\mathfrak{D}e_i$ are integers. Since this can be repeated for each i, $1 \leq i \leq n$, and $(\delta, \mathfrak{D})^{-1}$ is also in $N_{\mathrm{Aff}(\mathbb{R}^n)}(E_{n,k})$ this means that $\mathfrak{D} \in \mathrm{Gl}(n, \mathbb{Z})$.
Analogously for (a, A):

$$\begin{aligned}(\delta, \mathfrak{D})(a, A)(\delta, \mathfrak{D})^{-1} &= (\delta + \mathfrak{D}a, \mathfrak{D}A)(-\mathfrak{D}^{-1}\delta, \mathfrak{D}^{-1}) \\ &= (\delta + \mathfrak{D}a - \mathfrak{D}A\mathfrak{D}^{-1}\delta, \mathfrak{D}A\mathfrak{D}^{-1}).\end{aligned}$$

To obtain an element of $E_{n,k}$, $\mathfrak{D}A\mathfrak{D}^{-1}$ should be equal to A. This implies that $\mathfrak{D}$ must be of the form $\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$ where $D_1 \in \mathrm{Gl}(k, \mathbb{Z})$ and $D_2 \in \mathrm{Gl}(n-k, \mathbb{Z})$. This is however not sufficient. The translational part

$$\delta + \mathfrak{D}a - A\delta = (\frac{1}{2}d_{11}, \ldots, \frac{1}{2}d_{k1}, 2x_{k+1}, \ldots, 2x_n)^t$$

must be equal to $(\frac{1}{2} + k_1, k_2, \ldots, k_n)^t$ for some $k_i \in \mathbb{Z}$. We conclude that $x_1, \ldots, x_k \in \mathbb{R}$, $x_{k+1}, \ldots x_n \in \frac{1}{2}\mathbb{Z}$, $d_{11} \in 1 + 2\mathbb{Z}$ and $d_{21}, \ldots d_{k1} \in 2\mathbb{Z}$. $\square$

In order to construct Anosov diffeomorphisms on $M_{n,k}$, it suffices to choose a pair $(\delta, \mathfrak{D})$ as in Lemma 4.4 with $\mathfrak{D}$ hyperbolic. Note that for each $n \geq 4$ and for any k with $2 \leq k \leq n - 2$ such a $\mathfrak{D}$ indeed exists. One can always construct a blocked diagonal matrix using the blocks $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix}$.

In the sequel we need the following technical lemma.

Lemma 4.5.

1. *For every hyperbolic* $H_2 \in \mathrm{Gl}(2,\mathbb{R})$ *with* $\det(H_2) = \pm 1$*, we have*

$$\operatorname{sgn}(\det(I_2 - H_2)) \neq \operatorname{sgn}(\det(I_2 + H_2)).$$

2. *For every integer* $n \geq 3$ *there exist hyperbolic* $H_n, H_n' \in \mathrm{Gl}(n,\mathbb{Z})$ *such that*

$$\begin{aligned} \operatorname{sgn}(\det(I_n - H_n)) &\neq \operatorname{sgn}(\det(I_n + H_n)); \\ \operatorname{sgn}(\det(I_n - H_n')) &= \operatorname{sgn}(\det(I_n + H_n')). \end{aligned}$$

Proof:

1. Suppose $H_2 = (h_{ij}) \in \mathrm{Gl}(2,\mathbb{R})$ is hyperbolic and $\det(H_2) = \pm 1$. Then

$$\begin{aligned} \det(I_2 - H_2) &= 1 - (h_{11} + h_{22}) + \det(H_2), \\ \det(I_2 + H_2) &= 1 + (h_{11} + h_{22}) + \det(H_2) \end{aligned}$$

and the eigenvalues of H_2 are given by

$$\frac{1}{2}\left(h_{11} + h_{22} \pm \sqrt{(h_{11} + h_{22})^2 - 4\det(H_2)}\right).$$

Since $\det(H_2) = \pm 1$, we can distinguish two cases:

a) If $\det(H_2) = -1$, then $\det(I_2 - H_2) = -(h_{11} + h_{22})$ and $\det(I_2 + H_2) = h_{11} + h_{22}$. Thus their sign can only be equal if $h_{11} + h_{22} = 0$. But in that case the eigenvalues of H_2 are of absolute value 1, contradicting the hyperbolicity of H_2.

b) If $\det(H_2) = 1$, then $\det(I_2 - H_2) = 2 - (h_{11} + h_{22})$ and $\det(I_2 + H_2) = 2 + (h_{11} + h_{22})$. Their sign can only be equal if

$$-2 \leq h_{11} + h_{22} \leq 2.$$

But then $(h_{11} + h_{22})^2 - 4\det(H_2) \leq 0$ and so for the eigenvalues λ_1 and λ_2 of H_2 we obtain $\lambda_1 = \overline{\lambda_2}$. Then again the eigenvalues of H_2 are of absolute value 1, since their product must be equal to 1.

2. In order to prove the second part, we first construct hyperbolic H_2, H_3 and H_4 such that $\operatorname{sgn}(\det(I_k - H_k)) \neq \operatorname{sgn}(\det(I_k + H_k))$ $(k = 2, 3, 4)$ and hyperbolic H_3' and H_4' with $\operatorname{sgn}(\det(I_k - H_k')) =$

$\operatorname{sgn}(\det(I_k + H'_k))$ (for $k = 3$ or 4). For H_2 one can take any hyperbolic matrix in $\mathrm{Gl}(2,\mathbb{Z})$ by the first step of this lemma, e.g. $H_2 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, for the matrices H_3, H'_3, H_4 and H'_4 one can take:

$$H_3 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 2 & 0 & 3 \end{pmatrix} \quad H'_3 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{pmatrix},$$

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 3 \end{pmatrix} \quad H'_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

For every integer $n \geq 3$ we now construct H_n and H'_n as a blocked diagonal matrix in $\mathrm{Gl}(n,\mathbb{Z})$ consisting of blocks inside $\mathrm{Gl}(2,\mathbb{Z})$, $\mathrm{Gl}(3,\mathbb{Z})$ and $\mathrm{Gl}(4,\mathbb{Z})$. Indeed, in order to find a general H_n, we can then consider a blocked diagonal matrix consisting of one block H_k $(2 \leq k \leq 4)$ together with the right amount of blocks H'_3. On the other hand, H'_5 can be build using one block H_2 and one block H_3, H'_6 using two blocks H'_3, H'_7 using one H'_3 and one H'_4, ... □

Now we prove the following theorem

Theorem 4.6. *For each dimension $n > 4$ and for each integer k, satisfying $2 \leq k < n - 2$, there exists a flat manifold M with $b_1(M) = k$ and admitting Anosov diffeomorphisms f, g such that $N(f) \neq |L(f)|$ and $N(g) = |L(g)|$.*

Proof: We work with the flat manifold $M_{n,k}$ constructed in the beginning of this section and the $(\delta, \mathfrak{D}) = (\delta, \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}) \in N_{\mathrm{Aff}(\mathbb{R}^n)}(E_{n,k})$ obtained in Lemma 4.4. To prove our statement concerning the Nielsen and Lefschetz number of f and g we use Theorem 2.5. So we have to calculate $\det(I_n - \mathfrak{D})$ and $\det(I_n - A\mathfrak{D})$.

$$\begin{aligned} \det(I_n - \mathfrak{D}) &= \det \begin{pmatrix} I_k - D_1 & 0 \\ 0 & I_{n-k} - D_2 \end{pmatrix} \\ &= \det(I_k - D_1)\det(I_{n-k} - D_2) \end{aligned}$$

$$\begin{aligned} \det(I_n - A\mathfrak{D}) &= \det \begin{pmatrix} I_k - D_1 & 0 \\ 0 & I_{n-k} + D_2 \end{pmatrix} \\ &= \det(I_k - D_1)\det(I_{n-k} + D_2) \end{aligned}$$

Notice that $\det(I_n - \mathfrak{D}) = 0$ or $\det(I_n - A\mathfrak{D}) = 0$ only occurs if 1 or -1 is an eigenvalue, which is not possible for hyperbolic diffeomorphisms. Because of Lemma 4.5 and the fact that $n-k \geq 3$, we know that we can find suitable D_2 and D_2' such that in the first case $\operatorname{sgn}(\det(I_n - \mathfrak{D})) = -\operatorname{sgn}(\det(I_n - A\mathfrak{D}))$ and in the second case $\operatorname{sgn}(\det(I_n - \mathfrak{D}')) = \operatorname{sgn}(\det(I_n - A\mathfrak{D}'))$. So with δ, D_1 as in Lemma 4.4,

1. $D_2 = H_{n-k}$ as in Lemma 4.5 and $f : M_{n,k} \to M_{n,k}$ induced by $(\delta, \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix})$, we have that $|L(f)| \neq N(f)$;
2. $D_2' = H_{n-k}'$ as in Lemma 4.5 and $g : M_{n,k} \to M_{n,k}$ induced by $(\delta, \begin{pmatrix} D_1 & 0 \\ 0 & D_2' \end{pmatrix})$, we have that $|L(g)| = N(g)$.

□

Remark 4.7. *The first part of this theorem is a generalization of a theorem due to S. Kwasik and K.B. Lee ([34]) proving the existence of such Anosov diffeomorphisms in each even dimension.*

4.2.2 Flat manifolds with first Betti number equal to $n-2$

We first prove the following lemma concerning the first Betti number.

Lemma 4.8. *Suppose M is a flat n-dimensional manifold with first Betti number $b_1(M) = k > 0$. Assume that $E = \pi_1(M)$ is its fundamental group and*

$$T : F \to \mathrm{Gl}(n, \mathbb{Z})$$

is its holonomy representation. Then T is $\mathbb{Q}$-equivalent to a representation of the form

$$T' : F \to \mathrm{Gl}(n, \mathbb{Z}) : x \mapsto \begin{pmatrix} I_k & 0 \\ 0 & B(x) \end{pmatrix},$$

with $B(x) \in \mathrm{Gl}(n-k, \mathbb{Z})$, $\forall x \in F$.

Proof: To prove this, let F be generated by $x_1, x_2, \ldots x_l$ and let $T(x_i) = A_i$ $(1 \leq i \leq l)$. For any i we define the set $V_i = \{\overrightarrow{v} \in \mathbb{Q}^n | \overrightarrow{v} A_i = \overrightarrow{v}\}$ and take $V = A_1 \cap A_2 \cap \ldots \cap A_l$. We know that $b_1(M) = k$ if and only if

$$k = n - rank(A_1 - I_n, \ldots, A_l - I_n)$$

or $rank(A_1 - I_n, \ldots, A_l - I_n) = n - k$. So V is a k-dimensional rational vector space.
Let us now define $Z_V = \mathbb{Z}^n \cap V$. Then Z_V is a free abelian subgroup of $\mathbb{Z}^n$ of rank k and $\mathbb{Z}^n/Z_V$ is torsion-free. Therefore we can determine a new set of free generators for $\mathbb{Z}^n$ by first choosing k generators for Z_V and then adding $n-k$ elements such that their canonical projections generate $\mathbb{Z}^n/Z_V$.
With respect to such a set of generators, determined by a change of basis matrix $P \in \mathrm{Gl}(n, \mathbb{Z})$, we find that the holonomy representation is now given as

$$T_1 : F \to \mathrm{Gl}(n, \mathbb{Z}) : x \mapsto PT(x)P^{-1} = \begin{pmatrix} I_k & 0 \\ C(x) & B(x) \end{pmatrix}$$

with $B(x) \in \mathrm{Gl}(n-k, \mathbb{Z})$ and $C(x)$ a $(n-k) \times k$ matrix with integral entries. As F is a finite group and $\mathbb{Q}$ is a field of characteristic 0, we know that any representation of F over $\mathbb{Q}$ is fully reducible. In fact there exists a matrix $Q \in \mathrm{Gl}(n, \mathbb{Q})$, such that $T' = QT_1Q^{-1}$ is of the form

$$T' : F \to \mathrm{Gl}(n, \mathbb{Z}) : x \mapsto \begin{pmatrix} I_k & 0 \\ 0 & B(x) \end{pmatrix}.$$

This finishes the proof of the lemma. □

Let us now return to the case $b_1(M) = n-2$. The flat manifold $M_{n,2}$ with fundamental group $E_{n,2}$ generated by the translations $(e_1, I_n), \ldots, (e_n, I_n)$ and $(a_1, A_1) = (\begin{pmatrix} \frac{1}{2} \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I_{n-2} & 0 \\ 0 & -I_2 \end{pmatrix})$ is an example of a manifold such that $b_1(M_{n,2}) = n-2$. Moreover, because of Theorem 4.3, we know that this manifold admits Anosov diffeomorphisms. Concerning $M_{n,2}$ and $E_{n,2}$ we can prove the following proposition:

Proposition 4.9. *Let $n \geq 4$. Suppose M is a n-dimensional flat manifold with $b_1(M) = n-2$ and $\pi_1(M) = E$. Then M admits Anosov diffeomorphisms if and only if E is in the same $\mathbb{Q}$-class as $E_{n,2}$.*

Being in the same $\mathbb{Q}$-class means that the holonomy representations of E and $E_{n,2}$ are $\mathbb{Q}$-equivalent.

Proof: It is an immediate consequence of Theorem 4.3 and the fact that $M_{n,2}$ admits Anosov diffeomorphisms, that any flat manifold M whose

fundamental group is the same $\mathbb{Q}$-class as $E_{n,2}$ also admits Anosov diffeomorphisms.
Conversely assume that M admits Anosov diffeomorphisms. Because of Lemma 4.8, we know that the holonomy representation of E is $\mathbb{Q}$-equivalent to a representation of the form

$$T' : F \to \mathrm{Gl}(n,\mathbb{Z}) : x \mapsto \begin{pmatrix} I_{n-2} & 0 \\ 0 & B(x) \end{pmatrix}$$

with $B(x) \in \mathrm{Gl}(2,\mathbb{Z})$. Moreover, as the holonomy representation is faithful the representation $B : F \to \mathrm{Gl}(2,\mathbb{Z}) : x \mapsto B(x)$ has to be faithful too. There are only 10 $\mathbb{Q}$-classes of such representations (see [3]). From these 10 $\mathbb{Q}$-classes of representations, there are only two of them satisfying Theorem 4.3, the trivial one (for the trivial group) and the representation $\varphi : \mathbb{Z}_2 \to \mathrm{Gl}(2,\mathbb{Z})$, mapping the non-identity element of $\mathbb{Z}_2$ onto $-I_2$. We have to exclude the trivial group F, for in this case M is a torus and has $b_1(M) = n$. Therefore, the only possibility is that the holonomy group of M is $\mathbb{Z}_2$ and its holonomy representation is $\mathbb{Q}$-equivalent to that of $M_{n,2}$. □

Now we can prove the theorem stated at the beginning of this section.

Theorem 4.10. *Suppose $n \geq 4$ and let M be a n-dimensional flat manifold with $b_1(M) = n-2$. Then, for each Anosov diffeomorphism $f : M \to M$:*

$$N(f) \neq |L(f)|.$$

Proof: Let $E = \pi_1(M)$ be the fundamental group of M. By Proposition 4.9, we know that the holonomy group of M is $\mathbb{Z}_2$. So E is generated by n translations and an element (a, A), where $A \in \mathrm{Gl}(n,\mathbb{Z})$ is of order two. Again by Proposition 4.9, we know that there exists a matrix $P \in \mathrm{Gl}(n,\mathbb{Q})$ such that $PAP^{-1} = \begin{pmatrix} I_{n-2} & 0 \\ 0 & -I_2 \end{pmatrix}$.
Now, any Anosov diffeomorphism of M is homotopic to an affine diffeomorphism induced by an element $(\delta, \mathfrak{D}) \in \mathrm{Aff}(\mathbb{R}^n)$ where $\mathfrak{D}$ is hyperbolic and $(\delta, \mathfrak{D})$ normalizes E inside $\mathrm{Aff}(\mathbb{R}^n)$. As a consequence, $\mathfrak{D}$ must normalize the linear part of E inside $\mathrm{Gl}(n,\mathbb{R})$. This is however equivalent to $\mathfrak{D}$ commuting with A, or PDP^{-1} commuting with PAP^{-1}. It is easy to see that in this case $P\mathfrak{D}P^{-1}$ is of the form

$$P\mathfrak{D}P^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$$

where $D_1 \in \mathrm{Gl}(n-2,\mathbb{Q})$, $D_2 \in \mathrm{Gl}(2,\mathbb{Q})$. Both D_1 and D_2 are hyperbolic, because $\mathfrak{D}$ is hyperbolic. Moreover, as the characteristic polynomial of $\mathfrak{D}$ has integral coefficients and unit constant term, the characteristic polynomials of D_1 and D_2 also have integral coefficients and therefore also unit constant terms. In particular $\det(D_2) = \pm 1$.

Let us now make the following computations:

$$\det(I_n - \mathfrak{D}) = \det(I_n - P\mathfrak{D}P^{-1}) = \det(I_{n-2} - D_1)\det(I_2 - D_2)$$

$$\det(I_n - A\mathfrak{D}) = \det(I_n - PAP^{-1}P\mathfrak{D}P^{-1}) = \det(I_{n-2} - D_1)\det(I_2 + D_2)$$

Lemma 4.5 implies then that $\mathrm{sgn}(\det(I_n - \mathfrak{D})) \neq \mathrm{sgn}(\det(I_n - A\mathfrak{D}))$. (Notice that $\det(I_{n-2} - D_1) \neq 0$, since $\mathfrak{D}$ is hyperbolic.) Since the Nielsen and Lefschetz number are homotopic invariant we have that $N(f) \neq |L(f)|$. $\square$

Remark 4.11. *The condition that f is a Anosov diffeomorphism, is necessary. If f is for example the affine diffeomorphism on the manifold $M_{n,2}$ induced by $(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} I_{n-2} & 0 \\ 0 & D_1 \end{pmatrix})$ with $D_1 \in \mathrm{Gl}(2,\mathbb{Z})$, then $\det(I_n - \mathfrak{D}) = \det(I_n - A\mathfrak{D}) = 0$. (For this f we have that $L(f) = N(f) = 0$.)*

4.3 Primitive flat manifolds

From now onwards, we concentrate on the class of primitive flat manifolds. These are the flat manifolds with $b_1(M) = 0$.

4.3.1 Primitive flat manifolds in dimension $n > 6$

We will work with the Bieberbach group $E_{k,l,m}$ which is generated by

$$(e_i, I_n)\ ,\ (a, A) = (a, \begin{pmatrix} I_k & 0 & 0 \\ 0 & -I_l & 0 \\ 0 & 0 & -I_m \end{pmatrix})$$

$$\text{and } (b, B) = (b, \begin{pmatrix} -I_k & 0 & 0 \\ 0 & I_l & 0 \\ 0 & 0 & -I_m \end{pmatrix})$$

with $n = k + l + m$, 1 in the i-th place and 0 everywhere else for e_i $(1 \leq i \leq n)$, $a = (\frac{1}{2}\ 0 \cdots 0\ \frac{1}{2}\ 0 \cdots\ 0)^t$ and $b = (0 \cdots 0\ \frac{1}{2}\ 0 \cdots 0)^t$. (with $\frac{1}{2}$ on the first and $(k + l + 1)$-th place for a and $\frac{1}{2}$ on the $(k + 1)$-th place for b.) $E_{k,l,m}$ fits into the following short exact sequence:

$$1 \to \mathbb{Z}^n \to E_{k,l,m} \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

and $M_{k,l,m} = E_{k,l,m} \backslash \mathbb{R}^n$ is a flat manifold with $b_1(M_{k,l,m}) = 0$.

Remark 4.12. *The manifold $M_{k,l,m}$ used in this theorem is an arbitrary element of an interesting class of primitive $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-manifolds introduced by P. Cobb in [10].*

Using analogous computations as in section 4.2.1 we can prove the following lemma:

Lemma 4.13. *Suppose $(\delta, \mathfrak{D}) \in \mathrm{Aff}(\mathbb{R}^n)$ and $n = k + l + m$. Then $(\delta, \mathfrak{D}) \in N_{\mathrm{Aff}(\mathbb{R}^n)}(E_{k,l,m})$ if and only if $(\delta, \mathfrak{D})$ is of one of the following forms:*

1. $$\delta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathfrak{D} = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix}$$

   with $x_1, \ldots, x_n \in \frac{1}{2}\mathbb{Z}$;

2. $$\delta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathfrak{D} = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & 0 & D_2 \\ 0 & D_3 & 0 \end{pmatrix}$$

   with $x_1, x_{k+1}, x_{k+l+1} \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ and the other $x_i \in \frac{1}{2}\mathbb{Z}$;

3. $$\delta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathfrak{D} = \begin{pmatrix} 0 & 0 & D_1 \\ 0 & D_2 & 0 \\ D_3 & 0 & 0 \end{pmatrix}$$

   with $x_{k+1} \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ and the other $x_i \in \frac{1}{2}\mathbb{Z}$;

4. $$\delta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathfrak{D} = \begin{pmatrix} 0 & D_1 & 0 \\ 0 & 0 & D_2 \\ D_3 & 0 & 0 \end{pmatrix}$$

   with $x_1 \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ and the other $x_i \in \frac{1}{2}\mathbb{Z}$;

5.
$$\delta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \textit{and} \quad \mathfrak{D} = \begin{pmatrix} 0 & 0 & D_1 \\ D_2 & 0 & 0 \\ 0 & D_3 & 0 \end{pmatrix}$$

with $x_1, x_{k+1} \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ *and the other* $x_i \in \frac{1}{2}\mathbb{Z}$*;*

6.
$$\delta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \textit{and} \quad \mathfrak{D} = \begin{pmatrix} 0 & D_1 & 0 \\ D_2 & 0 & 0 \\ 0 & 0 & D_3 \end{pmatrix}$$

with $x_{k+l+1} \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ *and the other* $x_i \in \frac{1}{2}\mathbb{Z}$ *.*

where in each case $D_1 \in \mathrm{Gl}(k, \mathbb{Z})$*,* $D_2 \in \mathrm{Gl}(l, \mathbb{Z})$*,* $D_3 \in \mathrm{Gl}(m, \mathbb{Z})$ *and for each* D_i *the first column is of the following form: the first entry is an odd integer, while the other entries are even.*

With this lemma and Lemma 4.5 we can prove in a completely analogue way as Theorem 4.6 the following theorem:

Theorem 4.14. *For each dimension* $n > 6$ *there exists a primitive flat manifold* M *admitting Anosov diffeomorphisms* f, g *such that* $N(f) \neq |L(f)|$ *and* $N(g) = |L(g)|$*.*

Remark 4.15. *The theorem holds for every element* $M_{k,l,m}$ *of the class of* $(\mathbb{Z}_2 \times \mathbb{Z}_2)-$*manifolds introduced by Cobb which admits Anosov diffeomorphisms. Notice that it is an easy consequence of Theorem 4.3 that* $M_{k,l,m}$ *admits Anosov diffeomorphisms if and only if* $k \geq 2, l \geq 2$ *and* $m \geq 2$ *(see also [41]).*

4.3.2 Primitive flat manifolds in dimension 6

In this section we prove:

Theorem 4.16. *Let* M *be a* 6*-dimensional primitive flat manifold which admits Anosov diffeomorphisms. Then for each Anosov diffeomorphism* $f : M \to M$*:*

$$N(f) = |L(f)|.$$

C. Cid and T. Schulz present in [9] a complete enumeration of all six dimensional Bieberbach groups E with $b_1(E) = 0$. There are 5004 groups divided into 24 families and for each family they also describe the $\mathbb{Q}$-decomposition of the holonomy representation. All the Bieberbach groups belonging to a given family have the same $\mathbb{Q}$- and $\mathbb{R}$-decomposition. Therefore, we can use Theorem 4.3 to find the suitable families containing Bieberbach groups which are fundamental groups of flat manifolds admitting Anosov diffeomorphisms. Notice that each one- and two-dimensional $\mathbb{Q}$-irreducible component of a representation T is also $\mathbb{R}$-irreducible. Going over the list of all families, one finds that there are only two suitable families (for the notation we refer to [9]):

1. **Family 1,1;1,1;1,1** (The corresponding holonomy representation decomposes into one-dimensional $\mathbb{Q}-$irreducible components, each with multiplicity 2.)
   In this family there are 4 Bieberbach groups E fitting into an exact sequence of the form
   $$1 \to \mathbb{Z}^6 \to E \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$
   and they all belong to the same $\mathbb{Q}$-class. Note that the group $E_{2,2,2}$ is one of the 4 Bieberbach groups belonging to this family.
2. **Family 2–1,2–1;1,1** (The corresponding holonomy representation decomposes into a one-dimensional and a two-dimensional $\mathbb{Q}-$irreducible component, each with multiplicity 2.)
   In this family there is one Bieberbach group E fitting into an exact sequence of the form
   $$1 \to \mathbb{Z}^6 \to E \to D_8 \to 1.$$

So in order to prove Theorem 4.16, we only have to deal with the 4 Bieberbach groups of the first family for which we treat the case of the group $E_{2,2,2}$ in detail and the Bieberbach group of the second family which we also treat in detail. The computations for the other 3 groups are similar.

Proposition 4.17. *Suppose $M_{2,2,2} = E_{2,2,2}\backslash\mathbb{R}^6$ and $f : M_{2,2,2} \to M_{2,2,2}$ is an Anosov diffeomorphism, then we have*

$$N(f) = |L(f)|.$$

<u>Proof:</u> Because of Lemma 4.2 and the fact that $L(f)$ and $N(f)$ are homotopy invariants we may assume that f is a hyperbolic diffeomorphism induced by a $(\delta, \mathfrak{D}) \in N_{\mathrm{Aff}(\mathbb{R}^6)}(E_{2,2,2})$. We can use Lemma 4.13

to give a full description of all possible lifts of hyperbolic affine diffeomorphisms on $M_{2,2,2}$. We find that there are six types of hyperbolic diffeomorphisms. We will use Theorem 2.5 and so we have to calculate, for each of the six types of hyperbolic diffeomorphism, the determinants $\det(I_6 - \mathfrak{D})$, $\det(I_6 - A\mathfrak{D})$, $\det(I_6 - B\mathfrak{D})$ and $\det(I_6 - AB\mathfrak{D})$ and check that they have the same sign.
The numbering in the computations correspond to those of Lemma 4.13.

1. Then

$$\begin{aligned}
\det(I_6 - \mathfrak{D}) &= \det(I_2 - D_1)\det(I_2 - D_2)\det(I_2 - D_3)\\
\det(I_6 - A\mathfrak{D}) &= \det(I_2 - D_1)\det(I_2 + D_2)\det(I_2 + D_3)\\
\det(I_6 - B\mathfrak{D}) &= \det(I_2 + D_1)\det(I_2 - D_2)\det(I_2 + D_3)\\
\det(I_6 - AB\mathfrak{D}) &= \det(I_2 + D_1)\det(I_2 + D_2)\det(I_2 - D_3)
\end{aligned}$$

   Because of Lemma 4.5 we have that these determinants all have the same sign.

2. Suppose $K = d_{13}d_{13} + d_{23}d_{32} + d_{14}d_{41} + d_{24}d_{42}$, then we have

$$\begin{aligned}
\det(I_6 - \mathfrak{D}) &= \det(I_2 - D_1)(1 + \det(D_2)\det(D_3) - K)\\
&= \det(I_6 - A\mathfrak{D}),\\
\det(I_6 - B\mathfrak{D}) &= \det(I_2 + D_1)(1 + \det(D_2)\det(D_3) + K)\\
&= \det(I_6 - AB\mathfrak{D})
\end{aligned}$$

   and the eigenvalues of $\begin{pmatrix} 0 & D_2 \\ D_3 & 0 \end{pmatrix}$ are

$$\pm\sqrt{\frac{K \pm \sqrt{K^2 - 4\det(D_2)\det(D_3)}}{2}}.$$

   Since D_2 and D_3 are elements of $\mathrm{Gl}(2,\mathbb{Z})$ we can distinguish two cases.

   a) Either $\det(D_2)\det(D_3) = -1$, so $\det(I_6 - \mathfrak{D}) = -K\det(I_2 - D_1)$ and $\det(I_6 - B\mathfrak{D}) = K\det(I_2 + D_1)$. Thus because of Lemma 4.5 the signs of these two determinants must be the same.
   b) Or $\det(D_2)\det(D_3) = 1$, so $\det(I_6 - \mathfrak{D}) = (2 - K)\det(I_2 - D_1)$ and $\det(I_6 - B\mathfrak{D}) = (2 + K)\det(I_2 + D_1)$. Thus because of Lemma 4.5 the signs of these two determinants are not equal if and only if $K = 0$ or $K = \pm 1$. But then the eigenvalues of $\mathfrak{D}$ are of absolute value one and this is not possible for a hyperbolic diffeomorphism.

3. and 6. Completely analogue to case 2, of course with another K.
4. and 5. In this case:

$$\det(I_6 - \mathfrak{D}) = \det(I_6 - A\mathfrak{D}) = \det(I_6 - B\mathfrak{D}) = \det(I_6 - AB\mathfrak{D}).$$

So for each type $(\delta, \mathfrak{D})$ we have that the four determinants have the same sign and because of Theorem 2.5 it follows that $N(f) = |L(f)|$ for each Anosov diffeomorphism f. $\square$

Remark 4.18. *The condition that f is a hyperbolic diffeomorphism is necessary. For example if f is induced by* $\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right)$ *we have that* $\det(I_6 - \mathfrak{D}) = 4$ *and* $\det(I_6 - A\mathfrak{D}) = -12$. *Hence $L(f) = -4$ and $N(f) = 8$. Use for instance Theorem 2.9 for the calculation of $L(f)$ and $N(f)$.*

Completely analogous computations can be done for the other three Bieberbach groups in family 1,1;1,1;1,1 For the single Bieberbach group in family 2–1,2–1;1,1 we have to work with the Bieberbach group E generated by

$$z_i = (e_i, I_6)\ ,\ (a, A) = \left(\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right)$$

$$\text{and} \qquad (b, B) = \left(\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \right).$$

In an analogue way as in the previous sections we first have to calculate $N_{\mathrm{Aff}(\mathbb{R}^n)}(E)$:

Lemma 4.19. *Suppose* $(\delta, \mathfrak{D}) \in \mathrm{Aff}(\mathbb{R}^6)$*,* $D_1 \in \mathrm{Gl}(4,\mathbb{Z})$ *and* $D_2 \in \mathrm{Gl}(2,\mathbb{Z})$*. Then* $(\delta, \mathfrak{D}) \in N_{\mathrm{Aff}(\mathbb{R}^n)}(E)$ *if and only if* $\delta = (x_1, \ldots, x_6)$*,* $\mathfrak{D} = (d_{ij}) = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$ *and the* x_i *and* d_{ij} *satisfies one of the following conditions:*

1. $x_1, x_2, x_3, x_4, x_6 \in \frac{1}{2}\mathbb{Z}$, $d_{56}, d_{66} \in \mathbb{Z}$, $d_{55} \in 1 + 4\mathbb{Z}$, $d_{65} \in 4\mathbb{Z}$,

a) $x_5 \in \frac{1}{2}\mathbb{Z}$ *and* $D_1 = \begin{pmatrix} d_{11} & d_{12} & 0 & -d_{12} \\ d_{21} & d_{22} & d_{21} & 0 \\ 0 & d_{12} & d_{11} & d_{12} \\ -d_{21} & 0 & d_{21} & d_{22} \end{pmatrix}$

with $d_{11}, d_{22} \in 1 + 2\mathbb{Z}$ *and* $d_{12}, d_{21} \in \mathbb{Z}$

b) $x_5 \in \frac{1}{8} + \frac{1}{2}\mathbb{Z}$ *and* $D_1 = \begin{pmatrix} d_{11} & d_{12} & d_{11} & 0 \\ 0 & d_{22} & d_{23} & d_{22} \\ -d_{11} & 0 & d_{11} & d_{12} \\ -d_{23} & -d_{22} & 0 & d_{22} \end{pmatrix}$

with $d_{12}, d_{23} \in 1 + 2\mathbb{Z}$ *and* $d_{11}, d_{22} \in \mathbb{Z}$

c) $x_5 \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ *and* $D_1 = \begin{pmatrix} 0 & d_{12} & d_{13} & d_{12} \\ d_{21} & 0 & -d_{21} & d_{24} \\ -d_{13} & -d_{12} & 0 & d_{12} \\ d_{21} & -d_{24} & d_{21} & 0 \end{pmatrix}$

with $d_{13}, d_{24} \in 1 + 2\mathbb{Z}$ *and* $d_{12}, d_{21} \in \mathbb{Z}$

d) $x_5 \in \frac{3}{8} + \frac{1}{2}\mathbb{Z}$ *and* $D_1 = \begin{pmatrix} d_{11} & 0 & -d_{11} & d_{14} \\ d_{21} & d_{22} & 0 & -d_{22} \\ d_{11} & -d_{14} & d_{11} & 0 \\ 0 & d_{22} & d_{21} & d_{22} \end{pmatrix}$

with $d_{14}, d_{21} \in 1 + 2\mathbb{Z}$ *and* $d_{11}, d_{22} \in \mathbb{Z}$

2. $x_1, x_2, x_3, x_4, x_6 \in \frac{1}{2}\mathbb{Z}$, $d_{56}, d_{66} \in \mathbb{Z}$, $d_{55} \in 3 + 4\mathbb{Z}$, $d_{65} \in 4\mathbb{Z}$,

a) $x_5 \in \frac{1}{2}\mathbb{Z}$ *and* $D_1 = \begin{pmatrix} 0 & d_{12} & d_{13} & d_{12} \\ d_{21} & d_{22} & d_{21} & 0 \\ d_{13} & d_{12} & 0 & -d_{12} \\ d_{21} & 0 & -d_{21} & -d_{22} \end{pmatrix}$

with $d_{13}, d_{22} \in 1 + 2\mathbb{Z}$ *and* $d_{12}, d_{21} \in \mathbb{Z}$

b) $x_5 \in \frac{1}{8} + \frac{1}{2}\mathbb{Z}$ *and* $D_1 = \begin{pmatrix} d_{11} & d_{12} & d_{11} & 0 \\ d_{21} & d_{22} & 0 & -d_{22} \\ d_{11} & 0 & -d_{11} & -d_{12} \\ 0 & -d_{22} & -d_{21} & -d_{22} \end{pmatrix}$

with $d_{12}, d_{21} \in 1 + 2\mathbb{Z}$ *and* $d_{11}, d_{22} \in \mathbb{Z}$

c) $x_5 \in \frac{1}{4} + \frac{1}{2}\mathbb{Z}$ *and* $D_1 = \begin{pmatrix} d_{11} & d_{12} & 0 & -d_{12} \\ d_{21} & 0 & -d_{21} & d_{24} \\ 0 & -d_{12} & -d_{11} & -d_{12} \\ -d_{21} & d_{24} & -d_{21} & 0 \end{pmatrix}$
with $d_{11}, d_{24} \in 1 + 2\mathbb{Z}$ *and* $d_{12}, d_{21} \in \mathbb{Z}$

d) $x_5 \in \frac{3}{8} + \frac{1}{2}\mathbb{Z}$ *and* $D_1 = \begin{pmatrix} d_{11} & 0 & -d_{11} & d_{14} \\ 0 & d_{22} & d_{23} & d_{22} \\ -d_{11} & d_{14} & -d_{11} & 0 \\ d_{23} & d_{22} & 0 & -d_{22} \end{pmatrix}$
with $d_{14}, d_{23} \in 1 + 2\mathbb{Z}$ *and* $d_{11}, d_{22} \in \mathbb{Z}$

Now we can prove the following proposition:

Propositie 4.20. *Suppose* $M = E\backslash\mathbb{R}^6$ *and* $f : M \to M$ *is a hyperbolic diffeomorphism induced by a* $(\delta, \mathfrak{D}) \in \mathrm{Aff}(\mathbb{R}^6)$, *then we have*

$$N(f) = |L(f)|.$$

Proof: As in proposition 4.17 we will use theorem 2.5 to prove the statement and so we have to calculate the eight determinants for each type $(\delta, \mathfrak{D})$ of lemma 4.19. In an analogue way as in proposition 4.17 we can prove that the sign of the determinants is equal for hyperbolic $\mathfrak{D}$. (Again we find that if the signs are not equal, $\mathfrak{D}$ must have an eigenvalue of absolute value one.) □

Remark 4.21. *The condition that* f *is a hyperbolic diffeomorphism, is necessary. For if* f *is induced by* $\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 3 \end{pmatrix} \right)$ *we have that* $\det(I_6 - D) = -16$ *and* $\det(I_6 - BD) = 16$. *(For this* f *we have* $L(f) = 4$ *and* $N(f) = 12$.*)*

As an immediate consequence of Propositions 4.17 and 4.20, we conclude that Theorem 4.16 holds.

In the following table we summarize the results obtained in this chapter. Note that the $f : M \to M$ in this table stands for an Anosov diffeomorphism of the n-dimensional flat Riemannian manifolds M mentioned above and that we indicate if there exists an f which satisfies the relation.

$b_1(M)$	n	$N(f) = \|L(f)\|$	$N(f) \neq \|L(f)\|$	Proof
$b_1(M) = 0$	$n = 6$	always	never	Theorem 4.16
$b_1(M) = 0$	$n > 6$	exists	exists	Theorem 4.14
$2 \leq b_1(M) < n - 2$	$n > 4$	exists	exists	Theorem 4.6
$b_1(M) = n - 2$	$n \geq 4$	never	always	Theorem 4.10
$b_1(M) = n$	$n \geq 1$	always	never	see [1]

Chapter 5

Infra-nilmanifolds with cyclic holonomy group

In this chapter we examine the infra-nilmanifolds with cyclic holonomy group.[4] From the observations of Anosov we already know that in general the Anosov theorem does not hold for such manifolds. Indeed, the Klein bottle is a flat manifold with $\mathbb{Z}_2$ as holonomy group. However, in the second section we establish a sufficient condition for the Anosov relation to hold. More concretely, suppose that x_0 generates the holonomy group F of an infra-nilmanifold with cyclic holonomy group and $T : F \to \mathrm{Aut}(G)$ is the associated holonomy representation. Then we show that the Anosov theorem still holds in case -1 is not an eigenvalue of $T_*(x_0)$. So we extend the Anosov theorem to a new (and large class) of infra-nilmanifolds.

In the case of infra-nilmanifolds with odd order holonomy group of Chapter 3, -1 is never an eigenvalue of any of the matrices obtained from the holonomy representation, since these matrices are all of odd order. Therefore these manifolds also satisfy the above condition. However the situation is in general more delicate for the infra-nilmanifolds with cyclic holonomy group (of even order $2k$), since although if -1 is not an eigenvalue of $T_*(x_0)$, it can become an eigenvalue of some powers of $T_*(x_0)$.

In the third section we examine wether this condition is also necessary and we need to distinguish two cases. For flat manifolds with cyclic holonomy group we are able to show that the condition is also necessary and so for these manifolds the case is completely solved. On the other hand, in Chapter 3 we already gave an example of an infra-nilmanifold with holonomy group $\mathbb{Z}_2$ for which the Anosov theorem holds. One can

[4] The results of this chapter can also be found in [14].

easily verify that the generator of the holonomy group does not satisfy the condition mentioned above. This shows again that the validity of the Anosov theorem for infra-nilmanifolds is much more complicated then for flat manifolds.

5.1 Cyclic groups of matrices

As one might expect the results in this chapter depend heavily on the fact that we work with infra-nilmanifolds with cyclic holonomy group. To take advantage of this we need two lemmas concerning matrices of finite order (i.e. about cyclic groups of matrices). We leave the proof of these lemmas to the reader.

Lemma 5.1. *Let $B \in \mathrm{Gl}(n, \mathbb{R})$ be of order d and let $d_0, d_1, \ldots, d_t$ be the divisors of d such that $1 = d_0 < d_1 < \cdots < d_t = d$. Then there exists $n_0, n_1, \ldots, n_t \in \mathbb{N}$ and a $P \in \mathrm{Gl}(\mathrm{n}, \mathbb{R})$ such that $n_0 + n_1 + \cdots + n_t = n$ and*

$$PBP^{-1} = \begin{pmatrix} B_0 & 0 & \cdots & 0 \\ 0 & B_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_t \end{pmatrix}$$

with $B_i \in \mathrm{Gl}(n_i, \mathbb{R})$ and B_i only has eigenvalues of order exactly d_i $(0 \leq i \leq t)$.

Of course, by B being of order d, we mean that $B^d = I_n$, and an eigenvalue λ of order d_i indicates that $\lambda^{d_i} = 1$. Note that certain n_i from the lemma might be 0.

Lemma 5.2. *Let $d > 0$ be an integer with divisors $d_0, d_1, \ldots, d_t$ such that $1 = d_0 < d_1 < \cdots < d_t = d$. Suppose $B = \begin{pmatrix} B_0 & 0 & \cdots & 0 \\ 0 & B_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_t \end{pmatrix}$ with $B_i \in \mathrm{Gl}(n_i, \mathbb{R})$ and B_i only has eigenvalues of order d_i $(0 \leq i \leq t)$. Let $n = n_0 + \cdots + n_t$ and suppose $C \in M_n(\mathbb{R})$ such that $CB = B^l C$ with $0 \leq l < d$. Then*

$$C = \begin{pmatrix} C_0 & 0 & \cdots & 0 \\ * & C_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & C_t \end{pmatrix}$$

with $C_i \in M_{n_i}(\mathbb{R})$ and $*$ indicates any block of real numbers $(0 \leq i \leq t)$.

Now suppose that M is an infra-nilmanifold with a cyclic holonomy group F, generated by an element x_0 and $T : F \to \mathrm{Aut}(G)$ is the associated holonomy representation. Assume that F, and so x_0, is of order $2^r k$ with $r \geq 0$ and k an odd integer. Let $f : M \to M$ be a continuous map and $(\delta, \mathfrak{D})$ be a homotopy lift of f.
Because of Theorem 1.10 we know that there exists an integer l, with $0 \leq l < 2^r k$, such that $T_*(x_0^l)\mathfrak{D}_* = \mathfrak{D}_* T_*(x_0)$. Therefore we can apply the previous lemmas to $T_*(x_0)$ and $\mathfrak{D}_*$. Suppose $d_0, d_1, \ldots, d_t$ are the divisors of $2^r k$ and suppose $1 = d_0 < d_1 < \cdots < d_t = 2^r k$. Because of Lemma 5.1 there exists $n_0, n_1, \ldots, n_t \in \mathbb{N}$ and a $P \in \mathrm{Gl(n,\mathbb{R})}$ such that $n_0 + n_1 + \cdots + n_t = n$ and

$$PT_*(x_0)P^{-1} = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & A_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_t \end{pmatrix}$$

with $A_i \in \mathrm{Gl}(n_i, \mathbb{R})$ and A_i only has eigenvalues of order d_i $(0 \leq i \leq t)$. Each d_i can be written as $2^s d$ with $s \geq 0$ and d an odd integer. Note that since $T_*(x_0)$ is of finite order, the only possible eigenvalues are ± 1 or non real eigenvalues with absolute value equal to one. Also note that any of the n_i can be zero.
Because of Lemma 5.2 we then have

$$P\mathfrak{D}_*P^{-1} = \begin{pmatrix} D_0 & 0 & \cdots & 0 \\ * & D_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & D_t \end{pmatrix}$$

with $D_i \in M_{n_i}(\mathbb{R})$ $(0 \leq i \leq t)$. We will use this notations throughout this chapter.
Finally the following lemma will be very useful.

Lemma 5.3. *Let $B, C \in M_n(\mathbb{R})$ be two real matrices such that $BC = CB$ and B has only non real eigenvalues. Then the multiplicity of any real eigenvalue of C must be even, which implies that $\det(I_n - C) \geq 0$.*

Proof: We prove this lemma by induction on n. Note that n is even because B only has non real eigenvalues.
Suppose $n = 2$ and λ is a real eigenvalue of C with eigenvector

v such that $Cv = \lambda v$. Then Bv is also an eigenvector of C, since $CBv = BCv = \lambda Bv$. Moreover, v and Bv are linearly independent over $\mathbb{R}$. Otherwise there would exist a $\mu \in \mathbb{R}$ such that $Bv = \mu v$ contradicting the fact that B has no real eigenvalues. So the dimension of the eigenspace of λ is 2 and therefore the multiplicity of λ must be 2. Suppose the lemma holds for $r \times r$ matrices with r even and $r < n$. We then have to show that the lemma holds for $n \times n$ matrices. Again, let λ be a real eigenvalue of C and v an eigenvector of C such that $Cv = \lambda v$. Then, for any $m \in \mathbb{N}$, we have that $B^m v$ is an eigenvector of C. Indeed, $CB^m v = B^m Cv = \lambda B^m v$. Let S be the subspace of $\mathbb{R}^n$ generated by all vectors $B^m v$ with $m \in \mathbb{N}$. Then, for any $s \in S$, we have that $Cs = \lambda s$, so S is part of the eigenspace of λ and secondly $Bs \in S$, which implies that S is a B-invariant subspace of $\mathbb{R}^n$. Let $\{v_1, \ldots, v_k\}$ be a basis for S, then we can complete this basis with $v_{k+1}, \ldots, v_n$ to obtain a basis for $\mathbb{R}^n$. Writing (the matrices of the linear transformations determined by) B and C with respect to this new basis, implies the existence of a matrix $P \in \mathrm{Gl}(n, \mathbb{R})$ such that

$$PCP^{-1} = \begin{pmatrix} \lambda I_k & C_2 \\ 0 & C_3 \end{pmatrix} \quad \text{and} \quad PBP^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$$

with B_1 a real $k \times k$ matrix; B_2, C_2 real $k \times (n-k)$ matrices; and B_3, C_3 real $(n-k) \times (n-k)$ matrices. Of course, the eigenvalues of B_1 and B_3 are also not real and $B_3 C_3 = C_3 B_3$. Therefore, k has to be even and we can proceed by induction on B_3 and C_3 to conclude that the real eigenvalues of C indeed have even multiplicities.

To prove the second claim of the lemma, we suppose that $\lambda_1, \ldots, \lambda_r$ are the real eigenvalues of C with even multiplicities $m_1, \ldots, m_r$ and that $\mu_1, \overline{\mu_1}, \ldots, \mu_t, \overline{\mu_t}$ are the complex eigenvalues of C with multiplicities $n_1, \ldots, n_t$. Then

$$\begin{aligned}
\det(I_n - C) &= (1-\lambda_1)^{m_1} \cdots (1-\lambda_r)^{m_r} \\
&\quad (1-\mu_1)^{n_1}(1-\overline{\mu_1})^{n_1} \cdots (1-\mu_t)^{n_t}(1-\overline{\mu_t})^{n_t} \\
&= (1-\lambda_1)^{m_1} \cdots (1-\lambda_r)^{m_r} \\
&\quad ((1-\mu_1)(\overline{1-\mu_1}))^{n_1} \cdots ((1-\mu_t)(\overline{1-\mu_t}))^{n_t} \\
&= (1-\lambda_1)^{m_1} \cdots (1-\lambda_r)^{m_r} |1-\mu_1|^{2n_1} \cdots |1-\mu_t|^{2n_t}
\end{aligned}$$

This last expression is clearly nonnegative since the m_i are even. $\square$

5.2 The Anosov theorem for infra-nilmanifolds with cyclic holonomy group

This section is completely devoted to the proof of the main result

Theorem 5.4. *Let M be an infra-nilmanifold with cyclic holonomy group F generated by x_0. Let $T : F \to \mathrm{Aut}(G)$ be the holonomy representation and suppose -1 is not an eigenvalue of $T_*(x_0)$. Then for any continuous map $f : M \to M$ we have that $N(f) = |L(f)|$.*

The proof of this theorem will be based on Theorem 2.5 and therefore we have to examine the sign of determinants $\det(I_n - T_*(x_0^m)\mathfrak{D}_*)$ for $0 \leq m < 2^r k$. Using the notations of the previous section we have that

$$\begin{aligned}\det(I_n - T_*(x_0^m)\mathfrak{D}_*) &= \det(I_n - PT_*(x_0^m)P^{-1}P\mathfrak{D}_*P^{-1}) \\ &= \det(I_{n_0} - A_0^m D_0)\cdots\det(I_{n_t} - A_t^m D_t)\end{aligned}$$

So it suffices to consider the determinants $\det(I_{n_i} - A_i^m D_i)$ separately. This allows us to reduce our investigation to the study of the sign of determinants of the form

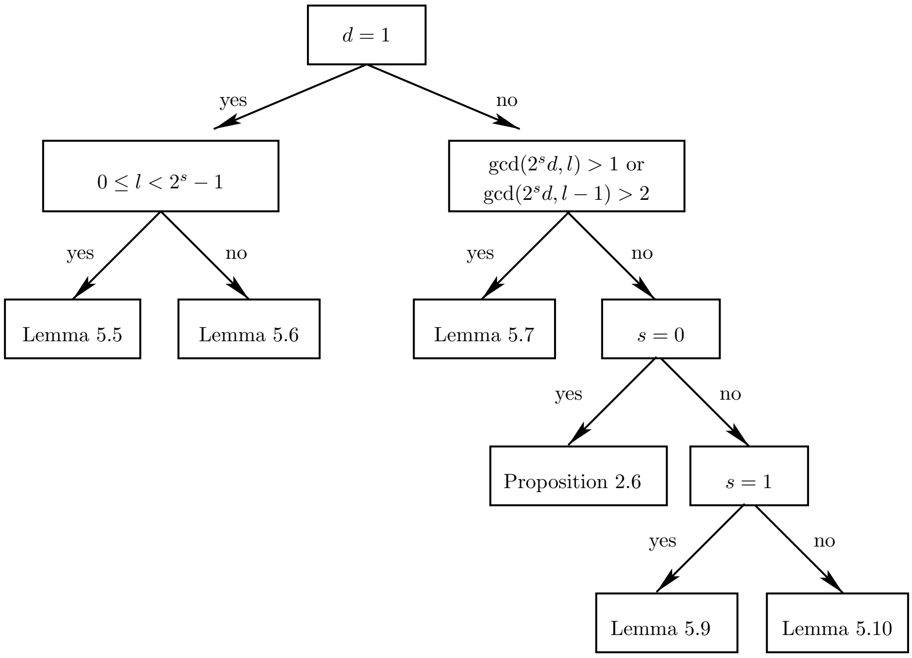
$$\det(I_n - A^m D), \quad 0 \leq m < 2^s d$$

where

1. $2^s d | 2^r k$ with d an odd integer,
2. $A^{2^s d} = I_n$,
3. each eigenvalue of A is **exactly** of order $2^s d$ and
4. $DA = A^l D$ for some l, with $0 \leq l < 2^s d$.

We will distinguish several cases, depending on the possible values of s, d and l. Note that we will not need the case $s = 1$ and $d = 1$ (which corresponds to an eigenvalue -1 in $T_*(x_0)$).

In order to deal with all possible cases, we have to prove a series of lemmas. In fact, we are going to use the following scheme in our treatment:



We start by looking at matrices of order 2^s (the left hand side of the scheme) and distinguish two cases depending on the value of l.

Lemma 5.5. *Suppose $A \in \mathrm{Gl}(n, \mathbb{R})$ only has eigenvalues of order 2^s with $s \geq 2$. Suppose $D \in M_n(\mathbb{R})$ such that $DA = A^l D$ with $0 \leq l < 2^s - 1$. Then the multiplicity of any real eigenvalue of D must be even and so we have that $\det(I_n - A^m D) \geq 0$ for any $0 \leq m < 2^s$.*

Proof: We prove this lemma by induction on s. Therefore we first look at the case where $s = 2$ and so l can be equal to $0, 1$ or 2. If $l = 0$ then $DA = D$ or $D(A - I_n) = 0_n$. Since 1 can not be an eigenvalue of A, we have that $A - I_n$ is invertible and so $D = 0_n$. Note that n is even since A only has non real eigenvalues and so in this case the lemma holds. If $l = 1$ we can apply Lemma 5.3 to A and D. Finally if $l = 2$ we have that $DA^2 = A^4 D = D$ and again $D = 0_n$ since $A^2 = -I_n$.

Assume now that $s > 2$ and that the lemma holds for smaller values of s. Since $DA = A^l D$ we also have that $DA^2 = (A^2)^l D$ and we can apply the induction hypothesis on A^2 which is of order 2^{s-1}. So we obtain for any l with $0 \leq l < 2^{s-1} - 1$ that the lemma holds, but the lemma holds already also for any l with $2^{s-1} \leq l < 2^s - 1$. Indeed for this l

we can consider $l' = l - 2^{s-1}$, then $0 \leq l' < 2^{s-1} - 1$ and we obtain $DA^2 = (A^2)^l D = (A^2)^{l'+2^{s-1}} D = (A^2)^{l'} D$. So, we can again apply the induction hypothesis.
There is still one case left, namely if $l = 2^{s-1} - 1$. Then we have that

$$DA^{2^{s-1}-1} = (A^{2^{s-1}-1})^{(2^{s-1}-1)} D = A^{(2^{s-1}-1)^2} D$$

Now $(2^{s-1}-1)^2 = 2^{2s-2} - 2 \cdot 2^{s-1} + 1 = 2^s(2^{s-2}-1) + 1$ and therefore $DA^{2^{s-1}-1} = AD$. This implies that $D(A + A^{2^{s-1}-1}) = (A + A^{2^{s-1}-1})D$. If we can show that $A + A^{2^{s-1}-1}$ only has non real eigenvalues then we can apply Lemma 5.3 to obtain that the lemma also holds in this case. Since A only has eigenvalues of order 2^s we know that $A + A^{2^{s-1}-1}$ only has eigenvalues of the form

$$e^{i \cdot \frac{2\pi t}{2^s}} + e^{i \cdot \frac{2\pi t}{2^s} \cdot (2^{s-1}-1)}$$

with $\gcd(t, 2^s) = 1$. Therefore t is odd and the imaginary part of such an eigenvalue is equal to

$$\begin{aligned}
\sin(\frac{2\pi t}{2^s}) + \sin(\frac{2\pi t}{2^s}(2^{s-1}-1)) &= \sin(\frac{2\pi t}{2^s}) + \sin(\pi t - \frac{2\pi t}{2^s}) \\
&= \sin(\frac{2\pi t}{2^s}) + \sin(\pi - \frac{2\pi t}{2^s}) \\
&= 2\sin(\frac{2\pi t}{2^s})
\end{aligned}$$

Note that $\frac{2\pi t}{2^s}$ can not be equal to $0, \pi$ or 2π since $\gcd(t, 2^s) = 1$ and therefore the eigenvalues of $A + A^{2^{s-1}-1}$ are always non real.

As is shown is Lemma 5.3, this implies also the second statement of the lemma. □
The second case is $l = 2^s - 1$ and there we obtain another result.

Lemma 5.6. *Suppose $A \in \mathrm{Gl}(n, \mathbb{R})$ only has eigenvalues of order 2^s with $s \geq 2$. Suppose $D \in M_n(\mathbb{R})$ such that $DA = A^{2^s-1}D$, then for any m, $0 \leq m < 2^s$,*

$$\det(I_n - D) = \det(I_n - A^m D)$$

Proof: Let $\mu_1, \overline{\mu_1}, \ldots \mu_w, \overline{\mu_w}$ be the different, non real eigenvalues of A with multiplicity $m_1, \ldots m_w$. A is diagonalizable so there exist a $Q \in \mathrm{Gl}(n, \mathbb{C})$ such that

$$QAQ^{-1} = \begin{pmatrix} \mu_1 I_{m_1} & 0 & \cdots & 0 & 0 \\ 0 & \overline{\mu_1} I_{m_1} & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \mu_w I_{m_w} & 0 \\ 0 & 0 & \cdots & 0 & \overline{\mu_w} I_{m_w} \end{pmatrix}$$

Since all the eigenvalues are of order 2^s this implies that

$$QA^{2^s-1}Q^{-1} = \begin{pmatrix} \overline{\mu_1} I_{m_1} & 0 & \cdots & 0 & 0 \\ 0 & \mu_1 I_{m_1} & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \overline{\mu_w} I_{m_w} & 0 \\ 0 & 0 & \cdots & 0 & \mu_w I_{m_w} \end{pmatrix}$$

One can easily verify that because of $DA = A^{2^{s-1}} D$ we obtain that

$$QDQ^{-1} = \begin{pmatrix} 0 & D_{i_1} & \cdots & 0 & 0 \\ D'_{i_1} & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & D_{i_w} \\ 0 & 0 & \cdots & D'_{i_w} & 0 \end{pmatrix}$$

with $D_{i_j}, D'_{i_j} \in M_{m_j}(\mathbb{C})$, $1 \leq j \leq w$. Then since the eigenvalues of A_i are roots of unity we can calculate $\det(I_n - A^m D)$ for any m, $0 \leq m < 2^s$.

$$
\begin{aligned}
&\det(I_n - A^m D)\\
&= \det(I_n - QA^mQ^{-1}QDQ^{-1})\\
&= \det\begin{pmatrix}
I_{m_1} & -(\mu_1)^m D_{i_1} & \cdots & 0 & 0\\
-(\overline{\mu_1})^m D'_{i_1} & I_{m_1} & \cdots & 0 & 0\\
\vdots & & \ddots & & \vdots\\
0 & 0 & \cdots & I_{m_w} & -(\mu_w)^m D_{i_w}\\
0 & 0 & \cdots & -(\overline{\mu_w})^m D'_{i_w} & I_{m_w}
\end{pmatrix}\\
&= \mu_1^{m\,m_1}\cdots\mu_w^{m\,m_w}\det\begin{pmatrix}
\overline{\mu_1}^m I_{m_1} & -D_{i_1} & \cdots & 0 & 0\\
-(\overline{\mu_1})^m D'_{i_1} & I_{m_1} & \cdots & 0 & 0\\
\vdots & & \ddots & & \vdots\\
0 & 0 & \cdots & \overline{\mu_w}^m I_{m_w} & -D_{i_w}\\
0 & 0 & \cdots & -(\overline{\mu_w})^m D'_{i_w} & I_{m_w}
\end{pmatrix}\\
&= (\mu_1\overline{\mu_1})^{m\,m_1}\cdots(\mu_w\overline{\mu_w})^{m\,m_w}\det\begin{pmatrix}
I_{m_1} & -D_{i_1} & \cdots & 0 & 0\\
-D'_{i_1} & I_{m_1} & \cdots & 0 & 0\\
\vdots & & \ddots & & \vdots\\
0 & 0 & \cdots & I_{m_w} & D_{i_w}\\
0 & 0 & \cdots & D'_{i_w} & I_{m_w}
\end{pmatrix}\\
&= \det(I_n - D)
\end{aligned}
$$

□

From now on we consider matrices of order $2^s d$ with $d > 1$. Again we will distinguish several cases depending on the value of l. In a first case we consider l for which $\gcd(2^s d, l) > 1$ or $\gcd(2^s d, l-1) > 2$.

Lemma 5.7. *Let $d > 1$ be an odd integer and let $s \geq 0$. Suppose $A \in \mathrm{Gl}(n,\mathbb{R})$ only has eigenvalues of order $2^s d$ and suppose $D \in M_n(\mathbb{R})$ such that $DA = A^l D$ with $0 \leq l < 2^s d$.*
If $\gcd(2^s d, l) > 1$ or $\gcd(2^s d, l-1) > 2$ then we have that $\det(I_n - A^m D) \geq 0$ for any m.

Proof: Assume first that $\gcd(2^s d, l) = l_1 > 1$ and that $l = l_1 l_2$ and $2^s d = l_1 d'$. Then we have

$$
\begin{aligned}
DA^{d'} &= (A^{d'})^l D\\
&= A^{2^s d l_2} D\\
&= D
\end{aligned}
$$

Now 1 is not an eigenvalue of $A^{d'}$ since $d' < 2^s d$ and the eigenvalues of A are of order $2^s d$. This implies as before that $D = 0_n$ and $\det(I_n - A^m D) = 1 \geq 0$ for any m.

Secondly, assume that $\gcd(2^s d, l-1) = l_1 > 2$ and that $l-1 = l_1 l_2$ and $2^s d = l_1 d'$. Then we have that

$$\begin{aligned} DA^{d'} &= (A^{d'})^l D \\ &= A^{d'(1+l_1 l_2)} D \\ &= A^{d'+2^s d l_2} D \\ &= A^{d'} D \end{aligned}$$

Now $A^{d'}$ only has non real eigenvalues since $d' < 2^{s-1}d$ and the eigenvalues of A are of order $2^s d$. So we can apply Lemma 5.3 to finish the proof of this lemma. $\square$

Finally, we are led to the situation in which we have to consider those l for which $\gcd(2^s d, l) = 1$ and $\gcd(2^s d, l-1) \leq 2$. If $s = 0$, then we can apply Theorem 3.3 since d is odd. On the other hand if $s \geq 1$, then $\gcd(2^s d, l-1) = 2$, since $\gcd(2^s d, l) = 1$ implies that l is odd.
The following lemma is useful to solve the second case.

Lemma 5.8. *Let $d > 1$ be an odd integer and let $s \geq 1$. Suppose $A \in \mathrm{Gl}(n, \mathbb{R})$ only has eigenvalues of order $2^s d$ and suppose $D \in M_n(\mathbb{R})$ such that $DA = A^l D$ with $0 \leq l < 2^s d$.*
If $\gcd(2^s d, l-1) = 2$ then we have that $\det(I_n - D) = \det(I_n - A^{2m} D)$ and $\det(I_n - AD) = \det(I_n - A^{2m+1} D)$ for any m.

Proof: We can establish for any m the following relations between the determinants.

$$\begin{aligned} \det(I_n - D) &= \det(A^m - DA^m) \det(A^{-m}) \\ &= \det(A^{-m}) \det(A^m - A^{ml} D) \\ &= \det(I_n - A^{m(l-1)} D) \end{aligned}$$

Since $\gcd(l-1, 2^s d) = 2$, we have that the group generated by A^{l-1} is of order $2^{s-1}d$ and thus consists of all even powers of A. It follows that $\det(I_n - D) = \det(I_n - A^{2m} D)$ for any m.
The second part can be proved analogously with $D' = AD$. $\square$

With this last lemma, we can prove the following lemma for $s = 1$.

Lemma 5.9. *Let $d > 1$ be an odd integer. Suppose $A \in \mathrm{Gl}(n, \mathbb{R})$ only has eigenvalues of order $2d$ and suppose $D \in M_n(\mathbb{R})$ such that $DA = A^l D$ with $0 \leq l < 2d$.*
If $\gcd(2d, l) = 1$ and $\gcd(2d, l-1) = 2$, then $\det(I_n - A^m D_i) \geq 0$ for any m or $\det(I_n - A^m D_i) \leq 0$ for any m.

Proof: Because of Lemma 5.8 we only have to prove the lemma for $\det(I_n - D)$ and $\det(I_n - AD)$. Moreover since d is odd and the eigenvalues of A^d are of order 2, it suffices to prove the lemma for $\det(I_n - D)$ and $\det(I_n - AD) = \det(I_n - A^d D) = \det(I_n + D)$.
Since $\gcd(2d, l) = 1$, Euler's theorem tells us that $l^{\phi(2d)} \equiv 1 \bmod 2d$. This implies that

$$\begin{aligned} D^{\phi(2d)} A &= D^{\phi(2d)-1} A^l D \\ &= D^{\phi(2d)-2} A^{l^2} D^2 \\ &= \dots \\ &= A^{l^{\phi(2d)}} D^{\phi(2d)} \\ &= A D^{\phi(2d)} \end{aligned}$$

So we obtain that, because of Lemma 5.3, the multiplicity of the real eigenvalues of $D^{\phi(2d)}$ must be even.
Suppose that $\lambda_1, \dots, \lambda_v$ are the real, positive eigenvalues of $D^{\phi(2d)}$ each with even multiplicity $m_1, \dots, m_v$; that $\gamma_1, \dots, \gamma_{t_1}$ are the real, strictly negative eigenvalues of $D^{\phi(2d)}$ and that $\mu_1, \overline{\mu_1}, \dots, \mu_{t_2}, \overline{\mu_{t_2}}$ are the non real eigenvalues of $D^{\phi(2d)}$. The eigenvalues of D must be $\phi(2d)$-th roots of these eigenvalues of $D^{\phi(2d)}$. Note that $\phi(2d)$ is an even integer since $d > 1$ and therefore a $\phi(2d)$-th root of γ_i or μ_k is always non real. While the $\phi(2d)$-th root of the λ_j can be real or non real , say α_j. But if α_j is an eigenvalue of D, then $\overline{\alpha_j}$ also has to be an eigenvalue of D. Now $(\overline{\alpha_j})^{\phi(2d)} = \overline{\alpha_j^{\phi(2d)}} = \lambda_j$. So $\overline{\alpha_j}$ has to be a $\phi(2d)$-th root of the same λ_j. Since the m_j are even, this implies for each j, $1 \leq j \leq v$, that the number of real eigenvalues of D coming from λ_j must be even.
Let us for each j, $1 \leq j \leq v$, denote the positive real $\phi(2d)$-th root by δ_j (and the negative real root by $-\delta_j$). We denote the multiplicity of δ_j, resp. $-\delta_j$, as an eigenvalue of D by r_j, resp. s_j. It is of course possible that δ_j or $-\delta_j$ is not an eigenvalue of D. In this case we take its multiplicity to be equal to 0. We then always have that $r_j + s_j \in 2\mathbb{Z}$.

Using the same arguments as in the proof of Lemma 5.3 and using the above information we know that the only factors that matter are

$$(1-\delta_1)^{r_1} \cdots (1-\delta_v)^{r_v} (1+\delta_1)^{s_1} \cdots (1+\delta_v)^{s_v}$$

in $\det(I_n - D)$ and

$$(1+\delta_1)^{r_1} \cdots (1+\delta_v)^{r_v} (1-\delta_1)^{s_1} \cdots (1-\delta_v)^{s_v}$$

in $\det(I_n + D)$.
For each $i \in \{1, \ldots, v\}$ there is in $\det(I_n - D)$ a factor of the form $(1-\delta_i)^{r_i}(1+\delta_i)^{s_i}$ and in $\det(I_n + D)$ there is a factor of the form $(1+\delta_i)^{r_i}(1-\delta_i)^{s_i}$. Suppose that $r_i \geq s_i$ (the other case is completely similar). Then

$$\begin{aligned}(1-\delta_i)^{r_i}(1+\delta_i)^{s_i} &= (1-\delta_i)^{(r_i-s_i)}(1-\delta_i)^{s_i}(1+\delta_i)^{s_i}\\ &= (1-\delta_i)^{r_i-s_i}(1-\delta_i^2)^{s_i}\end{aligned}$$

and

$$(1+\delta_i)^{r_i}(1-\delta_i)^{s_i} = (1+\delta_i)^{r_i-s_i}(1-\delta_i^2)^{s_i}$$

Since $r_i + s_i \in 2\mathbb{Z}$, we have that $r_i - s_i$ is also an even integer. So in both cases the first factor is positive and the second factor is the same. This ends the proof of this lemma. $\square$

Finally we can prove the following lemma for $s \geq 2$.

Lemma 5.10. *Let $d > 1$ be an odd integer and $s \geq 2$. Suppose $A \in \mathrm{Gl}(n, \mathbb{R})$ only has eigenvalues of order $2^s d$ and suppose $D \in M_n(\mathbb{R})$ such that $DA = A^l D$ with $0 \leq l < 2^s d$.*
If $\gcd(2^s d, l-1) = 2$ and $l \not\equiv 2^s - 1 \bmod 2^s$, then $\det(I_n - A^m D) \geq 0$ for any m.
If $\gcd(2^s d, l-1) = 2$ and $l \equiv 2^s - 1 \bmod 2^s$, then $\det(I_n - D) = \det(I_n - A^m D)$ for any m.

Proof: Because of Lemma 5.8 we only have to prove the lemma for $\det(I_n - D)$ and $\det(I_n - AD)$. Lemma 5.8 also implies that $\det(I_n - AD) = \det(I_n - A^d D)$ since d is odd. Now A^d only has eigenvalues of order 2^s and if $l \not\equiv 2^s - 1 \bmod 2^s$ we can apply Lemma 5.5 to D and A^d. If on the other hand $l \equiv 2^s - 1 \bmod 2^s$ then we can apply Lemma 5.6 to D and A^d. $\square$

With all these lemmas we can now prove the main result.
Proof of Theorem 5.4: Denote the order of F by $2^r k$ with k an odd integer. Let $(\delta, \mathfrak{D})$ be a homotopy lift of f and suppose that $\mathfrak{D}_* T_*(x_0) = T_*(x_0^l)\mathfrak{D}_*$. Suppose $d_0, d_1, \ldots, d_t$ are the divisors of $2^r k$ and suppose $1 = d_0 < d_1 < \cdots < d_t = 2^r k$. Because of Lemma 5.1 and the condition on $T_*(x_0)$ there exists $n_0, n_2, \ldots, n_t \in \mathbb{N}$ and a $P \in \mathrm{Gl}(\mathrm{n}, \mathbb{R})$ such that $n_0 + n_2 + \cdots + n_t = n$ and

$$PT_*(x_0)P^{-1} = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_t \end{pmatrix}$$

with $A_i \in \mathrm{Gl}(n_i, \mathbb{R})$ and A_i only has eigenvalues of order d_i. $(0 \leq i \leq t)$. Note that $n_1 = 0$ since -1 is not an eigenvalue of $T_*(x_0)$. Because of Lemma 5.2 we also have

$$P\mathfrak{D}_*P^{-1} = \begin{pmatrix} D_0 & 0 & \cdots & 0 \\ * & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & D_t \end{pmatrix}$$

with $D_i \in M_{n_i}(\mathbb{R})$ $(0 \leq i \leq t)$ and $*$ can be any block of real numbers. If we want to use Theorem 2.5, we have to calculate

$$\det(I_n - T_*(x_0^m)\mathfrak{D}_*) = \det(I_{n_0} - A_0^m D_0) \cdots \det(I_{n_t} - A_t^m D_t)$$

for any m, $0 \leq m < 2^r k$. As explained before, we consider all the factors above separately (so we fix an i and see what happens when m varies). If $d_i = 2^s$ $(s \geq 2)$ then we can, depending on l, apply Lemma 5.5 or Lemma 5.6 to show that all these factors have the same sign. Note that in case s is zero $(i = 0)$, $\det(I_{n_0} - A_0^m D_0)$ does not depend on m. By our assumption, the case $s = 1$ does not occur.
If d_i is not a power of 2 then we can, again depending on the value of l (see the scheme on page 67 and the discussion immediately before Lemma 5.8), apply Lemma 5.7, Theorem 3.3, Lemma 5.9 or Lemma 5.10 to show that these factors also have the same sign.
So in each case we obtain that the condition in Theorem 2.5 is satisfied and for each f we have $N(f) = |L(f)|$. $\square$

Remark 5.11.

- *The condition that -1 is not an eigenvalue of $T_*(x_0)$ is crucial since D. Anosov constructed a counterexample on the Klein Bottle which has $\mathbb{Z}_2$ as holonomy group. In the following section we go deeper into this.*
- *In the following chapter we prove that the Anosov theorem holds for orientable generalized Hantzsche-Wendt manifolds. This implies that it is not straight forward to generalize Theorem 5.4 to infra-nilmanifolds with other holonomy groups if -1 is an eigenvalue.*

In the following sections we examine the infra-nilmanifolds M with cyclic holonomy group, for which the holonomy representation does not satisfy the condition of Theorem 5.4. We have to distinguish two cases for these manifolds. In the case that M is a flat manifold, we are always able to construct a continuous map $f : M \to M$ such that $N(f) \neq |L(f)|$. So in the case of flat manifolds with cyclic holonomy group, we have a complete picture.
As already mentioned in the introduction, the example in section 3.3 shows that the same does not hold for infra-nilmanifolds in general.

5.3 The sharpness of the main result for flat manifolds

In order to construct a continuous map f on a flat manifold with cyclic holonomy group which does not satisfy the condition in Theorem 5.4, we already know that we only have to work with orientable manifolds. Indeed, flat non-orientable manifolds always admit an expanding map f and so Theorem 3.6 implies that we have a counter example.
Note that, because of Proposition 1.6, the manifolds which satisfy the conditions of Theorem 5.4 are orientable.

For flat orientable manifolds we can prove the following proposition.

Proposition 5.12. *Let M be a n-dimensional, orientable, flat manifold with cyclic holonomy group F generated by x_0. Let $T : F \to \mathrm{Gl}(n, \mathbb{Z})$ be the associated holonomy representation and suppose -1 is an eigenvalue of $T_*(x_0)$.*
Then there always exists a continuous map $f : M \to M$ such that $N(f) \neq |L(f)|$.

<u>Proof:</u> As -1 is an eigenvalue of the generator of the holonomy representation, F has to be a cyclic group of even order, say $2m$.
$\pi_1(M)$ is a n-dimensional Bieberbach group with translation subgroup $Z \cong \mathbb{Z}^n$ and holonomy group $\mathbb{Z}_{2m} = \langle x_0 \rangle$, for some $m \geq 1$. $\pi_1(M)$ fits in a short exact sequence

$$1 \to Z \cong \mathbb{Z}^n \to \pi_1(M) \to \mathbb{Z}_{2m} = \langle x_0 \rangle \to 1. \tag{5.1}$$

This short exact sequence determines a faithful representation $\varphi : \mathbb{Z}_{2m} \to \mathrm{Aut}(Z)$ (when viewed as a real representation is actually the same as the holonomy representation T). With respect to a good choice of generators of the free abelian group Z, φ is represented by blocked diagonal matrices, with

$$\varphi(x_0) = \begin{pmatrix} A(x_0) & 0 \\ * & C(x_0) \end{pmatrix}$$

where $A(x_0)$ only has eigenvalues ± 1 and $C(x_0)$ has no real eigenvalues. It follows that $A(x_0)$ is a matrix of order 2 and by eventually changing our set of generators for Z again, we can assume that

$$\varphi(x_0) = \begin{pmatrix} -I_s & 0 & 0 \\ * & I_t & 0 \\ * & * & C(x_0) \end{pmatrix}.$$

for some integers $s, t \geq 0$.
Now, -1 is an eigenvalue of $\varphi(x_0)$ ($s \neq 0$) and we also assume that M is orientable, which means that s is even, thus at least 2. Therefore, we will write

$$\varphi(x_0) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -I_{s-2} & 0 & 0 \\ * & * & * & I_t & 0 \\ * & * & * & * & C(x_0) \end{pmatrix}.$$

The group $\pi_1(M)$ is determined by a 2-cocycle $f : \mathbb{Z}_{2m} \times \mathbb{Z}_{2m} \to \mathbb{Z}^n$. This means that the group $\pi_1(M) = \mathbb{Z}^n \times \mathbb{Z}_{2m}$ (as a set) and the product in $\pi_1(M)$ is given by

$$\forall z, z' \in \mathbb{Z}^n,\ \forall x, y \in \mathbb{Z}_{2m} :\ (z, x)(z', y) = (z + \varphi(x)z' + f(x, y), xy).$$

Any element of $H^2(\mathbb{Z}_{2m}, \mathbb{Z}^n)$ has an order dividing $2m$. Therefore, there exists a map $g : \mathbb{Z}_{2m} \to \mathbb{Z}^n$, with $\delta g = 2mf$. (Recall that this means that $\delta g(x, y) = \varphi(x)g(y) - g(xy) + g(x) = 2mf(x, y)$)
It is now easy to check that

$$\psi_1 : \pi_1(M) = \mathbb{Z}^n \times \mathbb{Z}_{2m} \to \mathrm{Aff}(\mathbb{R}^n) : (z, x) \mapsto (z + \frac{g(x)}{2m}, \varphi(x))$$

realizes the group $\pi_1(M)$ as an affine group, with its translation subgroup Z mapped isomorphically onto $\mathbb{Z}^n$. Let us consider the image of $(0, x_0)$:

$$\psi_1(0,x_0)=\left(\begin{pmatrix}\frac{x}{2m}\\ \frac{y}{2m}\\ \frac{u_1}{2m}\\ \frac{u_2}{2m}\\ \frac{u_3}{2m}\end{pmatrix},\begin{pmatrix}-1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -I_{s-2} & 0 & 0\\ * & * & * & I_t & 0\\ * & * & * & * & C(x_0)\end{pmatrix}\right).$$

(For some $x,y\in\mathbb{Z}$, $u_1\in\mathbb{Z}^{s-2}$, $u_2\in\mathbb{Z}^t$, $u_3\in\mathbb{Z}^{n-s-t}$).
Let $v=(-\frac{x}{4m},-\frac{y}{4m},0,0,0)^t\in\mathbb{R}^n$ and take $\psi_2=(v,I_n)\psi_1(v,I_n)^{-1}$. Then, $\psi_2(z,1)=\psi_1(z,1)$ for all $z\in\mathbb{Z}^n$ and

$$\psi_2(0,x_0)=\left(\begin{pmatrix}0\\ 0\\ \frac{u_1}{2m}\\ \frac{u_2}{2m}\\ \frac{u_3}{2m}\end{pmatrix},\begin{pmatrix}-1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -I_{s-2} & 0 & 0\\ * & * & * & I_t & 0\\ * & * & * & * & C(x_0)\end{pmatrix}\right).$$

There exists a rational matrix

$$P=\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & I_s & 0 & 0\\ * & * & * & I_t & 0\\ * & * & * & 0 & I_{n-s-t}\end{pmatrix}\in \mathrm{Gl}(n,\mathbb{Q})$$

such that

$$P\varphi(x_0)P^{-1}=\begin{pmatrix}-1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -I_{s-2} & 0 & 0\\ 0 & 0 & 0 & I_t & 0\\ 0 & 0 & 0 & * & C(x_0)\end{pmatrix}.$$

Let D_1 be the matrix

$$D_1=\begin{pmatrix}2 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & I_{s-2} & 0 & 0\\ 0 & 0 & 0 & I_t & 0\\ 0 & 0 & 0 & 0 & I_{n-s-t}\end{pmatrix}.$$

It is obvious that D_1 commutes with $P\varphi(x_0)P^{-1}$ and thus $D_2=P^{-1}D_1P\in\mathrm{Gl}(n,\mathbb{Q})$ commutes with $\varphi(x_0)$. The matrix D_2 is an invertible rational matrix, whose characteristic polynomial (which is the same as the characteristic polynomial of D_1) has integer coefficients and unit constant term. This implies, by a result of H. Porteous ([49]), that there exists a positive integer k such that $D_3 = D_2^k \in \mathrm{Gl}(n, \mathbb{Z})$. Now, D_3 has almost all eigenvalues equal to 1, except two positive real eigenvalues, say $\lambda_1 > 1$ and $\lambda_2 = \frac{1}{\lambda_1} < 1$. Again by taking a suitable power of D_3, we obtain a new matrix $D_4 = D_3^l$ (for some l) such that its two eigenvalues different from 1 are $\lambda_1^l, \frac{1}{\lambda_1^l}$ and satisfy

$$\lambda_1^l > 2m + 1 \Rightarrow \frac{1}{\lambda_1^l} < \frac{1}{2m+1}.$$

Now, finally, let $\mathfrak{D} = (2m+1)D_4$. It is obvious that $\mathfrak{D}$ still commutes with $\varphi(x_0)$. The matrix $\mathfrak{D}$ is of the form

$$\mathfrak{D} = \begin{pmatrix} a\ b & 0 & 0 & 0 \\ b\ c & 0 & 0 & 0 \\ 0\ 0\ (2m+1)I_{s-2} & 0 & 0 & \\ *\ * & * & (2m+1)I_t & 0 \\ *\ * & * & 0 & (2m+1)I_{n-s-t} \end{pmatrix}.$$

Each $*$ indicates a block with entries in $(2m+1)\mathbb{Z}$ and the block $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ has two positive real eigenvalues μ_1 and μ_2 satisfying: $\mu_1 > (2m+1)^2$ and $\mu_2 < 1$.

Now, conjugating with $(0, \mathfrak{D})$ inside $\mathrm{Aff}(\mathbb{R}^n)$ induces an endomorphism of $\psi_2(E)$. This follows from the following observations:

1. Let $(z, I_n) \in \psi_2(E)$ (so $z \in \mathbb{Z}^n$), $(0, \mathfrak{D})(z, I_n)(0, \mathfrak{D}^{-1}) = (\mathfrak{D}z, I_n) \in \psi_2(E)$.
2. We compute the image of $\psi_2(0, x_0) = (t(x_0), \varphi(x_0))$, with $t(x_0) = (0, 0, \frac{u_1}{2m}, \frac{u_2}{2m}, \frac{u_3}{2m})^t$:

$$\begin{aligned} (0, \mathfrak{D})\psi_2(0, x_0) &= (0, \mathfrak{D})(t(x_0), \varphi(x_0)) \\ &= (\mathfrak{D}t(x_0), \mathfrak{D}\varphi(x_0)) \\ &= ((\mathfrak{D} - I_n)t(x_0), I_n)(t(x_0), \varphi(x_0))(0, \mathfrak{D}) \end{aligned}$$

   By the construction of $\mathfrak{D}$ and the fact that the first 2 entries of $t(x_0)$ are zero, we have that $(\mathfrak{D} - I_n)t(x_0) \in \mathbb{Z}^n$. This implies that $(0, \mathfrak{D})(t(x_0), \varphi(x_0))(0, \mathfrak{D})^{-1} \in \psi_2(E)$.

Let f be the map on M, induced by conjugation with $(0, \mathfrak{D})$. On the one hand, we have that

$$\det(I_n - \mathfrak{D}) = \underbrace{(1-\mu_1)}_{<0}\underbrace{(1-\mu_2)}_{>0}\underbrace{(-2m)^{s-2}}_{>0}(-2m)^t\underbrace{(-2m)^{n-s-t}}_{>0},$$

while on the other hand

$$\begin{aligned}&\det(I_n - \varphi(x_0)\mathfrak{D})\\ &= \underbrace{(1+\mu_1)}_{>0}\underbrace{(1+\mu_2)}_{>0}\underbrace{(2+2m)^{s-2}}_{>0}(-2m)^t\underbrace{\det(I_{n-s-t}-(2m+1)C(x_0))}_{>0}.\end{aligned}$$

It is obvious that these two determinants differ in sign and Theorem 2.5 again implies that $N(f) \neq |L(f)|$.

□

Chapter 6

Generalized Hantzsche-Wendt manifolds

In this chapter we work with a special, but still large, class of flat manifolds, namely the generalized Hantzsche-Wendt manifolds. These are n-dimensional flat manifolds with holonomy group isomorphic to $\mathbb{Z}_2^{n-1}$. In dimension 2 the Klein bottle is the only example of such a manifold. In dimension 3 there are three examples, namely the classical Hantzsche-Wendt manifold, which is an orientable manifold, and two non-orientable manifolds. For a more detailed study of generalized Hantzsche-Wendt manifolds we refer to [26],[46], [47] and [51].

As shown in Chapter 3 we only have to concentrate on flat orientable generalized Hantzsche-Wendt manifolds, since for the non-orientable ones we already obtained that the Anosov theorem does not hold. Although we have a partial picture, there is still a large part missing since in [46] it is shown that the number of orientable Hantzsche-Wendt manifolds grows exponentially with the dimension.

Perhaps surprisingly, we are able to show that the Anosov theorem always holds in the class of orientable generalized Hantzsche-Wendt manifolds. [5] These manifolds are however 'totally' different from the infra-nilmanifolds discussed in the previous chapters. In general terms one can say that up till now we preferred the 'absence' of the eigenvalue -1 in the matrices obtained from the holonomy representation. Now we have the complete opposite, since in the first section we will show that the image of the corresponding holonomy representation, contains n matrices having -1 as an eigenvalue with (maximal!) multiplicity $n-1$.

[5] The results of this chapter can also be found in [15]

6.1 Definition and properties

As already mentioned, a n-dimensional flat manifold is called a generalized Hantzsche-Wendt (GHW) manifold if its holonomy group is isomorphic to $\mathbb{Z}_2^{n-1}$. In this case the Bieberbach group $E = \pi_1(M)$ is called a GHW group. A n-dimensional Bieberbach group E is said to be diagonal if its lattice $\mathbb{Z}^n$ of translations has an orthogonal basis for which the holonomy representation T diagonalizes. In [51] J.P. Rossetti and A. Szczepański proved the following theorem:

Theorem 6.1. *The fundamental group of a generalized Hantzsche-Wendt manifold is diagonal.*

Suppose M is a n-dimensional flat GHW manifold and $T : \mathbb{Z}_2^{n-1} \to \mathrm{Gl}(n, \mathbb{Z})$ is the associated holonomy representation. Because of Theorem 6.1 we may assume that $T(x)$ is diagonal for each $x \in \mathbb{Z}_2^{n-1}$ and hence we know that the diagonal elements must be 1 or -1.
If moreover, M is an orientable manifold, then for each $x \in \mathbb{Z}_2^{n-1}$, the diagonal entries of $T(x)$ consist of an even number of -1's while the others are 1. In fact it is obvious that in $\mathrm{Gl}(n, \mathbb{Z})$ there are exactly 2^{n-1} diagonal matrices whose diagonal entries consist of an even number of -1's while the other entries are 1. It follows that

Corollary 6.2. *Let M be a n-dimensional orientable flat GHW manifold and $T : \mathbb{Z}_2^{n-1} \to \mathrm{Gl}(n, \mathbb{Z})$ its associated holonomy representation. Then*

1. *The image of $T : \mathbb{Z}_2^{n-1} \to \mathrm{Gl}(n, \mathbb{Z})$ is completely determined;*
2. *n is an odd integer;*
3. *the first Betti number of M is 0.*

Proof: Note that, because of Theorem 6.1, T is a diagonal representation. Since a holonomy representation must be faithful and the holonomy group is of order 2^{n-1}, the first result follows immediately.
Suppose n is an even integer, then there exist a $x \in \mathbb{Z}_2^{n-1}$ such that $T(x) = -Id$ which is not possible since $\pi_1(M)$ is torsion-free.
Using the techniques introduced in Chapter 4 one can verify that the first Betti number of an orientable flat GHW manifold must be 0. □

In an analogous way, one can consider the class of non-orientable flat GHW manifolds M. For these manifolds, it is easy to prove that the first Betti number of M must be 0 or 1. In the latter case again the image of the holonomy representation is completely determined. In the former case there are more possibilities. In [51] the authors show that there are $n/2$ possibilities for n even and $(n+1)/2$ possibilities for n odd. For more details we refer to [46] and [51].

To finish this section, we already note the following for flat GHW manifolds.

Theorem 6.3. *Let M be a flat GHW manifold and let $f : M \to M$ be a self-homotopy equivalence of M, then $N(f) = L(f)$.*

Proof: In [51], J.P.Rossetti and A. Szczepański proved that the outer automorphism group $\mathrm{Out}(\pi_1(M))$ of the fundamental group of a flat GHW manifold is finite. Therefore it follows from [42] that each self-homotopy equivalence is homotopically periodic. And because of [34], we obtain that $N(f) = L(f)$. □

6.2 Orientable flat GHW manifolds

The goal of this section is to prove the Anosov theorem for orientable flat GHW manifolds M, i.e. to show that the relation $N(f) = |L(f)|$ holds for any continuous map $f : M \to M$ (Theorem 6.10).

Let M be an orientable n-dimensional flat GHW manifold with fundamental group $E = \pi_1(M)$ and associated holonomy representation $T : F \to \mathrm{Gl}(n, \mathbb{Z})$. It's image $T(F)$ is exactly the set of the 2^{n-1} diagonal matrices with 1 or -1 on the diagonal and such that the number of -1's is even (and the number of 1's is odd, as the dimension of M must be odd, Corollary 6.2).
The group $T(F)$ is hence generated by the set of diagonal matrices A_i $(1 \leq i \leq n-1)$, where all the diagonal entries are -1, except the entry on the i-th row and column, which is 1. As a consequence, we can assume that the group E is generated by

$$(z_1, I_n), \ldots, (z_n, I_n), (a_1, A_1), \ldots (a_{n-1}, A_{n-1}) \tag{6.1}$$

where $z_i \in \mathbb{Z}^n$ $(1 \leq i \leq n)$ and $a_i \in \mathbb{R}^n$ are appropriate translational parts. Let us, for the rest of this chapter, denote the k-th component of an element $b \in \mathbb{R}^n$ by b^k.
R. Miatello and J.P. Rossetti showed in [47, Lemma 1.4] that we can assume that the elements a_i^k of a_i are 0 or $\frac{1}{2}$. Although this result does not allow us to specify the translational parts a_i completely, we

do already know that a_i^i must be $\frac{1}{2}$ for each i. Indeed, if $a_i^i = 0$, a simple computation shows that $(a_i, A_i) \cdot (a_i, A_i) = (a_i + A_i a_i, A_i^2) =$

$$\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, I_n\right),$$ which would imply that E has torsion.

We will refer to a generating set (6.1) of E, with the $a_i^k = 0$ or $\frac{1}{2}$ as a suitable generating set.

Remark 6.4. *Let $A_n = A_1 \cdot A_2 \cdot \ldots \cdot A_{n-1}$. Then A_n is also a diagonal matrix with all diagonal entries equal to -1, except the last one which is 1 (since n is odd). Because of [47, Lemma 1.4] we can assume that there exists $a_n \in \mathbb{R}^n$ with components 0 or $\frac{1}{2}$ such that $(a_n, A_n) \in E$.*

As already mentioned above, in order to prove the Anosov theorem for flat orientable GHW manifolds, it is sufficient to deal with the continuous maps of M which are induced by a suitable affine endomorphism of $\mathbb{R}^n$. Therefore we need a full description of all affine endomorphisms $g : \mathbb{R}^n \to \mathbb{R}^n$ which are obtained in Corollary 1.11.

Lemma 6.5. *Let M be an orientable n–dimensional flat GHW manifold ($n \geq 3$) and let $(z_1, I_1), \ldots, (z_n, I_n), (a_1, A_1), \ldots, (a_{n-1}, A_{n-1})$ be a suitable generating set of $\pi_1(M) = E$. Assume θ is a homomorphism of E and $(\delta, \mathfrak{D}) \in \mathbb{R}^n \rtimes M_n(\mathbb{R}^n)$ is a suitable affine endomorphism (i.e. $\forall \alpha \in E : \theta(\alpha) \cdot (\delta, \mathfrak{D}) = (\delta, \mathfrak{D}) \cdot \alpha$). Denoting the (i, j)-th entry of $\mathfrak{D}$ by d_{ij} we then have the following:*

1. *If there exists a $j \in \{1, 2, \ldots, n\}$ such that $\theta(a_j, A_j) = (z, I_n)$, with $z \in \mathbb{Z}^n$ (the image of some (a_j, A_j) is a pure translation), then for all i, $(1 \leq i \leq n)$, $d_{ik} = 0$ if $k \neq j$ $(1 \leq k \leq n)$, while d_{ij} is an even integer.*
2. *If there exists a $j \in \{1, 2, \ldots, n\}$ such that $\theta(a_j, A_j) = (z, I_n)(b, B)$, with $z \in \mathbb{Z}^n$, $b \in \mathbb{R}^n$ a translation consisting of 0's and $\frac{1}{2}$'s and $B \neq I_n$ a finite product of A_i's (the image of some (a_j, A_j) is not a pure translation), then there exists a i $(1 \leq i \leq n)$ such that d_{ij} is an odd integer and $d_{ik} = 0$ for all $k \neq j$ $(1 \leq k \leq n)$.*

Proof:

1. Since $\theta(a_j, A_j) \cdot (\delta, \mathfrak{D}) = (z, I_n) \cdot (\delta, \mathfrak{D}) = (z + \delta, \mathfrak{D})$ and $(\delta, \mathfrak{D}) \cdot (a_j, A_j) = (\delta + \mathfrak{D}a_j, \mathfrak{D}A_j)$, it follows that $\mathfrak{D} = \mathfrak{D}A_j$ and $z + \delta = \delta + \mathfrak{D}a_j$.
   Now, A_j only has $+1$ in the j-th column while the other diagonal entries are -1, this forces all the columns of $\mathfrak{D}$ to be zero, except the j-th column.
   For the translational parts we must have that $z+\delta = \delta+\mathfrak{D}a_j$. Since $a_j^j = \frac{1}{2}$ and only the j-th column of $\mathfrak{D}$ is non-zero, it follows that $(z^1 + d^1, \ldots, z^n + d^n)^t = (d^1 + \frac{1}{2}d_{1j}, \ldots, d^n + \frac{1}{2}d_{nj})^t$. So d_{ij} must be an even integer for all i $(1 \leq i \leq n)$.
2. B is a finite product of A_i's, so B is also a diagonal matrix with 1's and -1's as diagonal entries. As above we denote the (i, j)-th entry of B by b_{ij}. Note that $(b, B) \cdot (b, B) = (b + Bb, I_n)$, from which it follows that $b+Bb$ can not be equal to zero. This implies that there exists a i $(1 \leq i \leq n)$ such that $b_{ii} = 1$ and $b^i = \frac{1}{2}$.
   Since $\theta(a_j, A_j) \cdot (\delta, \mathfrak{D}) = (z + b, B) \cdot (\delta, \mathfrak{D}) = (z + b + B\delta, B\mathfrak{D})$ and $(\delta, \mathfrak{D}) \cdot (a_j, A_j) = (\delta + \mathfrak{D}a_j, \mathfrak{D}A_j)$, we have that $B\mathfrak{D} = \mathfrak{D}A_j$ and $z + b + B\delta = \delta + \mathfrak{D}a_j$.
   Since $b_{ii} = 1$, the i-th row of $B\mathfrak{D}$ equals the i-th row of $\mathfrak{D}$. Similarly the j-th column of $\mathfrak{D}A_j$ equals the j-th column of $\mathfrak{D}$ while the other columns of $\mathfrak{D}A_j$ are equal to minus the corresponding column of $\mathfrak{D}$. It follows that $d_{ik} = 0$ for all $k \neq j$ $(1 \leq k \leq n)$.
   To show that d_{ij} is an odd integer, we look at the translational parts for which we know that $z + b + B\delta = \delta + \mathfrak{D}a_j$. Again, using $b_{ii} = 1$, the i-th component of the above equality reduces to
   $$z^i + b^i + d^i = d^i + \sum_{k=1}^{n} d_{ik}a_j^k = d^i + d_{ij}a_j^j = d^i + \frac{1}{2}d_{ij}.$$
   Since $b^i = \frac{1}{2}$ this shows that d_{ij} must be an odd integer.

□

Using the lemma above, we can now prove the following proposition in which we describe the linear parts of all suitable affine endomorphisms of $\mathbb{R}^n$.

Proposition 6.6. *Let M be an orientable n-dimensional flat GHW manifold with fundamental group $\pi_1(M) = E$ $(n \geq 3)$. Let θ be a homomorphism of E and $(\delta, \mathfrak{D}) \in \mathbb{R}^n \rtimes M_n(\mathbb{R}^n)$ be a suitable affine endomorphism. Then*

1. *$\mathfrak{D}$ is either the zero $n \times n$ matrix 0_n or*
2. *$\mathfrak{D}$ is an element of $\mathrm{Gl}(n, \mathbb{Q})$ such that in each row and each column of $\mathfrak{D}$ there is exactly one non-zero element, which is an odd integer.*

Proof: To prove this proposition we distinguish three cases depending on the number of (a_i, A_i)'s which are mapped onto a pure translation.

Case 1: Suppose that θ is a homomorphism of E such that two or more of the images of $(a_1, A_1), \ldots, (a_n, A_n)$ are pure translations. So there exist i and j, $i \neq j$, for which $\theta(a_i, A_i)$ and $\theta(a_j, A_j)$ are pure translations. Then Lemma 6.5 implies that $\mathfrak{D} = 0_n$.

Case 2: Suppose that θ is a homomorphism of E such that just one of the images of $(a_1, A_1), \ldots, (a_n, A_n)$ is a pure translation. So part one of Lemma 6.5 implies that all the elements of $\mathfrak{D}$ are even integers. But there also has to exist a j such that $\theta(a_j, A_j)$ is not a pure translation. So part two of Lemma 6.5 implies that there is a i such that d_{ij} is an odd integer. Since this gives a contradiction, we conclude that no such homomorphism exists.

Case 3: Suppose that θ is a homomorphism of E such that none of the images of $(a_1, A_1), \ldots (a_n, A_n)$ is a pure translation. In this situation, we can determine $\mathfrak{D}$ completely. Namely, since $\theta(a_1, A_1)$ is not a pure translation, Lemma 6.5 implies that there exists a i_1 such that $d_{i_1 1}$ is odd, while the other elements of the i_1-th row are zero. Doing the same for (a_2, A_2) we obtain a i_2 such that $d_{i_2 2}$ is odd, while the other elements of the i_2-th row are zero. Clearly $i_2 \neq i_1$, otherwise we would have that $d_{i_2 2}$ is zero on the one hand and an odd integer on the other hand. This can be done for all the images of $(a_1, A_1), \ldots (a_{n-1}, A_{n-1})$ and (a_n, A_n), so we have completely determined $\mathfrak{D}$ as a matrix with exactly one non-zero, odd entry in each row and column.

□

Remark 6.7. *In the above proposition, we did not mention the image of the pure translations. But since the $\mathfrak{D}$ always consist of integers, one can easily show that the image of a pure translation under the homomorphism again must be a pure translation.*

Now that we have a clear view on the different possibilities for the linear parts of the suitable affine endomorphisms $(\delta, \mathfrak{D})$, we can start to use Theorem 2.5.

As a first step, in the following lemma we calculate the determinants which appear in Theorem 2.5 and in a second lemma we determine the signs of these determinants. To a $(n \times n)$-matrix $\mathfrak{D}$ with in each row and each column exactly one non-zero element (as in the second part of Proposition 6.6), one can associate an unique permutation μ of n elements. Namely, for any $i = 1, 2, \ldots, n$ let $\mu(i)$ be the unique index such that $d_{i\mu(i)} \neq 0$. Clearly μ is a permutation of n elements and μ has a unique cycle decomposition.

Lemma 6.8. *Suppose B is any diagonal matrix whose diagonal entries b_{ii} are 1's and -1's.*

1. *If $\mathfrak{D} = 0_n$ is the zero $(n \times n)$-matrix, then $\det(I_n - B\mathfrak{D}) = 1$ for all possible B.*
2. *Let $\mathfrak{D}$ be a $(n \times n)$-matrix, such that in each row and each column of $\mathfrak{D}$ there is exactly one non-zero element and let μ be the associated permutation. Let the cycle decomposition of μ be*

$$(l_1^1 \; l_2^1 \; \cdots \; l_{p_1}^1)(l_1^2 \; l_2^2 \; \cdots \; l_{p_2}^2) \cdots (l_1^r \; l_2^r \; \cdots \; l_{p_r}^r).$$

Then we have that

$$\begin{aligned}
&\det(I_n - B\mathfrak{D}) \\
&= \det(B) \times \prod_{i=1}^{r} (b_{l_1^i l_1^i} b_{l_2^i l_2^i} \cdots b_{l_{p_i}^i l_{p_i}^i} - d_{l_1^i \mu(l_1^i)} d_{l_2^i \mu(l_2^i)} \cdots d_{l_{p_i}^i \mu(l_{p_i}^i)}) \\
&= \det(B) \times \prod_{i=1}^{r} (b_{l_1^i l_1^i} b_{l_2^i l_2^i} \cdots b_{l_{p_i}^i l_{p_i}^i} - d_{l_1^i l_2^i} d_{l_2^i l_3^i} \cdots d_{l_{p_i}^i l_1^i}).
\end{aligned}$$

Proof:

1. Trivial.
2. Since $B^2 = I_n$, we have that $\det(I_n - B\mathfrak{D}) = \det(B). \det(B - \mathfrak{D})$. Now we show that

$$\det(B - \mathfrak{D}) = \prod_{i=1}^{r} (b_{l_1^i l_1^i} b_{l_2^i l_2^i} \cdots b_{l_{p_i}^i l_{p_i}^i} - d_{l_1^i \mu(l_1^i)} d_{l_2^i \mu(l_2^i)} \cdots d_{l_{p_i}^i \mu(l_{p_i}^i)}).$$

We do this by induction on n, the case $n = 1$ being trivial. Suppose the formula holds for $(n-1) \times (n-1)$-matrices $(n \geq 2)$, then we distinguish two cases:

a) $d_{nn} \neq 0$ or equivalently $\mu(n) = n$. Then $\det(B - \mathfrak{D}) = (b_{nn} - d_{n\mu(n)}) \det(B' - \mathfrak{D}')$, where B' (resp. $\mathfrak{D}'$) is obtained from the matrix B (resp. $\mathfrak{D}$) by deleting the last row and column. By applying the induction hypothesis to $B' - \mathfrak{D}'$ we obtain the result.

b) $d_{nn} = 0$. Then there exists a unique k such that $d_{nk} \neq 0$ (or $\mu(n) = k$) and a unique l with $d_{ln} \neq 0$ (or $\mu(l) = n$). So in the cycle decomposition of μ, there is a cycle of the form $(l_1 \cdots l\ n\ k \cdots\ l_p)$.
In the last column of the matrix $B - \mathfrak{D}$ there are two non-zero elements, namely b_{nn} and $-d_{ln}$.
Since $b_{nn} = \pm 1$ we can create a zero in the last column of the l-th row by adding to the l-th row $b_{nn} \cdot d_{ln}$ times the last row. In the computation below, this operation is used in the step indicated with $(*)$.
In this calculation, we again use B' and $\mathfrak{D}'$ to denote the matrices obtained by deleting the last row and column of B and $\mathfrak{D}$ respectively. Note that the l-th row of $\mathfrak{D}'$ and the k-th column of $\mathfrak{D}'$ only consists of zeros and that in each other row and each other column of $\mathfrak{D}'$ there is exactly one non-zero element.
Also, we need $\mathfrak{D}''$ obtained from $\mathfrak{D}'$ by changing the k-th component of the l-th row of $\mathfrak{D}'$ to $-b_{nn} d_{ln} d_{nk}$. Since $\mathfrak{D}'$ is from the above form, we then have that in each row and each column of $\mathfrak{D}''$ there is exactly one non-zero element. So

$$
\begin{aligned}
\det(B - \mathfrak{D}) &= \det \begin{pmatrix} & & & & & & 0 \\ & & & & & & \vdots \\ & & & & & & 0 \\ & & & B' - \mathfrak{D}' & & & -d_{ln} \\ & & & & & & 0 \\ & & & & & & \vdots \\ & & & & & & 0 \\ 0 & \cdots & 0 & -d_{nk} & 0 & \cdots\ 0 & b_{nn} \end{pmatrix} \\
&\overset{(*)}{=} \det \begin{pmatrix} & & & & & & 0 \\ & & & B' - \mathfrak{D}'' & & & \vdots \\ & & & & & & 0 \\ 0 & \cdots & 0 & -d_{nk} & 0 & \cdots\ 0 & b_{nn} \end{pmatrix} \\
&= b_{nn} \det(B' - \mathfrak{D}'')
\end{aligned} \tag{6.2}
$$

Now we can associate a permutation μ'' of $n-1$ elements to $\mathfrak{D}''$. The cycle decomposition of μ'' is obtained from the cycle decomposition of μ by replacing in this decomposition the cycle $(l_1 \cdots l\, n\, k \cdots l_p)$ which contains n, with the cycle $(l_1 \cdots l\, k \cdots l_p)$.
The induction hypothesis now applies to the matrix $B' - \mathfrak{D}''$ and it follows that all factors in the expansion of $\det(B' - \mathfrak{D}'')$ except the one containing the terms of the l-th row are of the desired form. The exceptional factor containing the elements of the l-th row and corresponding to the cycle $(l_1 \cdots l\, k \cdots l_p)$ is of the following form:

$$b_{l_1 l_1} \cdots b_{ll} \cdots b_{l_p l_p} - d_{l_1 \mu(l_1)} \cdots (b_{nn} d_{ln} d_{nk}) \cdots d_{l_p \mu(l_p)}$$

We can multiply the factor above with b_{nn} to get

$$b_{nn} b_{l_1 l_1} \cdots b_{ll} \cdots b_{l_p l_p} - b_{nn} d_{l_1 \mu(l_1)} \cdots (b_{nn} d_{ln} d_{nk}) \cdots d_{l_p \mu(l_p)}$$

which equals ($n = \mu(l)$ and $\mu(n) = k$)

$$b_{l_1 l_1} \cdots b_{ll} b_{nn} \cdots b_{l_p l_p} - d_{l_1 \mu(l_1)} \cdots d_{l \mu(l)} d_{n \mu(n)} \cdots d_{l_p \mu(l_p)}$$

This part of the expansion of (6.2) together with the other factors show that $\det(B - \mathfrak{D})$ is of the desired form.

$\square$

Note that the number of factors of the determinants in the previous lemma does not depend on B, but only on $\mathfrak{D}$ (in fact, only on the form of D determined by μ). So for a given $\mathfrak{D}$, with $\mathfrak{D}$ as in the second case of Lemma 6.5, and any diagonal matrix B consisting of 1's and -1's, we obtain that $\det(I_n - B\mathfrak{D}) = \det(B)(\pm 1 - x_1) \cdots (\pm 1 - x_k)$. Here $x_1, \ldots, x_k \in 1 + 2\mathbb{Z}$ and the ± 1's depend on the choice of B. In this perspective the following lemma is crucial.

Lemma 6.9. *Fix an integer $k \geq 1$ and $x_1, \ldots, x_k \in 1 + 2\mathbb{Z}$. Then*

either $(\epsilon_1 - x_1) \cdots (\epsilon_k - x_k) \geq 0$ *for all possible* $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$
or $(\epsilon_1 - x_1) \cdots (\epsilon_k - x_k) \leq 0$ *for all possible* $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$.

Proof: Suppose there exists $\epsilon_1, \ldots, \epsilon_k$ and $\epsilon'_1, \ldots, \epsilon'_k$ such that $(\epsilon_1 - x_1) \cdots (\epsilon_k - x_k) > 0$ and $(\epsilon'_1 - x_1) \cdots (\epsilon'_k - x_k) < 0$. This is only possible if there exist a j for which $1 - x_j > 0$ and $-1 - x_j < 0$ or conversely

$1 - x_j < 0$ and $-1 - x_j > 0$. But in the former case we quickly see that $x_j = 0$ which is not possible and in the latter case we obtain the contradiction that $x_j > 1$ and $x_j < -1$. □

We are now ready to prove the Anosov theorem for flat orientable GHW manifolds.

Theorem 6.10. *Let $n \geq 3$ be an odd integer and M a (flat) orientable n-dimensional generalized Hantzsche-Wendt manifold. Then for each continuous map $f : M \to M$ we have that $N(f) = |L(f)|$.*

Proof: Suppose $f : M \to M$ is a continuous map on M. Due to Corollary 1.11 we know that f is homotopic to a map g induced by an suitable affine endomorphism $(\delta, \mathfrak{D})$ of $\mathbb{R}^n$ and due to Proposition 6.6 we know how $\mathfrak{D}$ looks like. Since the Nielsen and Lefschetz number are homotopy invariants it suffices to prove the theorem for the map g. We use Theorem 2.5 to verify that $N(g) = |L(g)|$. Therefore we have to calculate $\det(I_n - T(x)\mathfrak{D})$ for each $x \in F$. Note that for each $x \in F$, $T(x)$ is a diagonal matrix whose diagonal entries consist of an even numbers of -1's while the others are 1 and so $\det(T(x)) = 1$. Therefore we can apply Lemma 6.8 and Lemma 6.9 to the determinants $\det(I_n - T(x)\mathfrak{D})$ which finishes the proof of the theorem. □

Part III

The Anosov theorem in small dimensions

In this part we verify what we know about the Anosov theorem for infra-nilmanifolds in small dimensions, i.e. up to dimension 4. In this way we show that many infra-nilmanifolds are already covered by the original result of Anosov and our results of the previous part. However there are still examples of infra-nilmanifolds which are not yet covered by the known theorems. By examining these manifolds we hope to find some tracks for further research, in order to close the remaining lack of theoretic results.

In a first chapter we examine the flat manifolds and we use [7] for a complete description of these flat manifolds. For flat manifolds we have more results which we can apply and the calculations are a lot easier. This results in a complete picture for dimension 3, but in dimension 4 only 55 of the 74 flat manifolds are covered. The 19 remaining manifolds can be divided in two classes. To be specific 4-dimensional flat manifolds with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and 4-dimensional flat manifolds with non-abelian holonomy group. In the last two sections we observe some intriguing results for these two classes.

In a second chapter we focus on infra-nilmanifolds up to dimension 4. We use [12] for a complete description of these infra-nilmanifolds. We also include the infra-nilmanifold introduced in [18] which is missing in the previous description. Firstly, the calculations for these manifolds are much more complicated than for flat manifolds. Therefore we start this chapter by formally showing how the calculations need to be done. Secondly we have less theoretic results which we can apply (for instance Proposition 5.12) and so lot of work had to be done (or many possibilities for further research are left over).

For all the manifolds which are not covered by the theoretic results, we present in this part a counterexample (in the case the Anosov theorem does not hold for the specific manifold) or a proof (in the opposite case). We demonstrate the calculations that need to be done throughout several well chosen examples. All this results in a complete answer to the question for which infra-nilmanifolds up to dimension 4 the Anosov theorem holds.

Chapter 7

Flat manifolds

In the sequel we need to refer explicitly to specific flat manifolds. As any flat manifold is determined by its fundamental group, this boils down to finding a way of explicitly referring to the associated Bieberbach group. We have chosen to use the numbers which can be found on "the page of low dimensional Bieberbach groups" at the CARAT website [7]. This allows us to say that the orientable GHW manifold corresponds to the number 10.1.1. Actually, this number refers to the $\mathbb{Z}$–class to which the fundamental group of the orientable GHW manifold belongs. Two Bieberbach groups can belong to the same $\mathbb{Z}$–class, by which one means that the corresponding integral holonomy representations are conjugated over $\mathbb{Z}$. If we need to talk about different Bieberbach groups in the same $\mathbb{Z}$–class, we will use the same order as on the CARAT website. E.g. when one asks the CARAT system for the Bieberbach groups of dimension 3 with holonomy group $\mathbb{Z}_2 \times \mathbb{Z}_2$, one receives as an answer:

```
#g2 % 1-th torsion free group in Z-class min.10.1.1
4   /2      % generator
 -2 0  0 0
  0 2  0 1
  0 0 -2 1
  0 0  0 2
4   /2      % generator
 2  0  0 1
 0 -2  0 0
 0  0 -2 0
 0  0  0 2
2^2  % order of the group
```

```
#g2 % 1-th torsion free group in Z-class min.9.1.1
4   /2      % generator
 -2 0 0 0
  0 2 0 0
  0 0 2 1
  0 0 0 2
4   /2      % generator
 2  0 0 1
 0 -2 0 0
 0  0 2 0
 0  0 0 2
2^2  % order of the group
#g2 % 2-th torsion free group in Z-class min.9.1.1
4   /2      % generator
 -2 0 0 0
  0 2 0 1
  0 0 2 1
  0 0 0 2
4   /2      % generator
 2  0 0 1
 0 -2 0 0
 0  0 2 0
 0  0 0 2
2^2  % order of the group
```

This table contains for each of the Bieberbach groups a set of generators, which should in any case be completed with a set of generators for the integral lattice $\mathbb{Z}^n$ (here $\mathbb{Z}^3$). The table above shows that the orientable GHW belongs to the $\mathbb{Z}$-class with number 10.1.1. while the two non-orientable ones both belong to the $\mathbb{Z}$–class 9.1.1.

Finally, we note here that in literature one can find other classifications of the Bieberbach groups and so implicitly also of the flat manifolds, see for instance [3]. In this book, Bieberbach groups are divided into families, families into crystal systems, ... and also finally into $\mathbb{Z}$–classes (before the really final step, up to isomorphism, of course).

Where useful, we will always mention these $\mathbb{Z}$-class numbers (and the exact number inside the $\mathbb{Z}$-class when needed). In this way, one can use [7] to find a precise description of the fundamental group of the respective manifold. We prefer to base our referring system on [7], instead of e.g. [3], because of the overall availability of [7].

7.1 General overview in dimension 3 and 4

For flat manifolds up to dimension 2, Anosov provided a complete picture, since we only have the circle in dimension 1 and the torus and the Klein bottle in dimension 2. By the theorems proved in the previous part of this thesis, we also get a complete overview of the situation in dimension 3, where there are 10 flat manifolds. We summarize in the following table.

Dimension 3				
Holonomy group	# manifolds	Orientable		Non orientable
		An th. holds	An.th. holds not	An.th. holds not
1	1	1	0	0
$\mathbb{Z}_2$	3	0	1	2
$\mathbb{Z}_3$	1	1	0	0
$\mathbb{Z}_4$	1	1	0	0
$\mathbb{Z}_6$	1	1	0	0
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	3	1	0	2
Total	10	5	1	4

We explain which information can be found in this table (and the corresponding one for flat manifolds in dimension 4 on page 98).

In the first column we list all possible holonomy groups and in the second column we give the number of corresponding manifolds. In the following columns we then summarize what we know, about the validity of the Anosov theorem for the specific manifolds. We do this by giving respectively the number of orientable manifolds for which the Anosov theorem holds; the number of orientable manifolds for which the Anosov theorem does not hold and the number of non-orientable manifolds (for which the Anosov theorem never holds because of Proposition 3.7).

We conclude that in dimension 3 there are 5 flat manifolds for which the Anosov theorem holds: the torus which is covered by Theorem 2.3, the 3 manifolds with cyclic holonomy group for which Theorem 5.4 holds and the orientable GHW manifold for which Theorem 6.10 holds.

From the 5 manifolds for which the Anosov theorem does not hold, there are 4 non-orientable manifolds for which Proposition 3.7 holds and one orientable manifold with holonomy group $\mathbb{Z}_2$ (7.1.1) for which Proposition 5.12 holds.

In contrary to dimension 3, in dimension 4 there are some manifolds which are not covered by the theorems of the previous chapters. From the 74 flat manifolds in dimension 4 there are 55 manifolds covered by the known results which mainly comes from Proposition 3.7 concerning non-orientable flat manifolds. So 19 manifolds still need to be examined by 'hand'. The results are summarized in the table below.

Dimension 4				
Holonomy group	# manifolds	Orientable		Non orientable
		An th. holds	An.th. holds not	An.th. holds not
1	1	1	0	0
$\mathbb{Z}_2$	5	0	2	3
$\mathbb{Z}_3$	2	2	0	0
$\mathbb{Z}_4$	6	2	0	4
$\mathbb{Z}_6$	4	1	0	3
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	12	0	0	12
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	26	4	5	17
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	1	0	0	1
$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	3	0	0	3
D_8	7	2	2	3
S_3	3	0	3	0
$\mathbb{Z}_2 \oplus S_3$	1	0	1	0
A_4	2	2	0	0
$\mathbb{Z}_2 \oplus A_4$	1	0	0	1
Total	74	14	13	47

Note that the first half of the table contains these manifolds for which we have a complete picture by the theorems of the previous part. Namely the torus, the 17 manifolds with cyclic holonomy group which are covered by Proposition 3.7, Theorem 5.4 and Proposition 5.12 and the 12 non-orientable GHW manifolds which are also covered by Proposition 3.7.

In the second half of the table we find 25 non-orientable manifolds for which we again can apply Proposition 3.7 to obtain that the Anosov theorem does not hold.

The remaining 19 manifolds which need a more detailed examination can be split up into two classes: 9 flat manifolds with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as holonomy group and 10 flat manifolds with a non-abelian holonomy group. Both of these classes are very interesting. The first class, for instance is exactly in between the class of flat manifolds with $\mathbb{Z}_2$ as holonomy group and the class of generalized Hantzsche-Wendt manifolds, two classes for which we already have a complete picture by Theorem 5.4, Proposition 5.12 and Theorem 6.10. Secondly in the previous chapters we mainly focused on infra-nilmanifolds with an abelian holonomy group (see Chapter 5 and 6, which explains why the second class could be interesting). In the following two sections we discuss the two classes in detail.

7.2 Flat manifolds in dimension 4 with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as holonomy group

Let M be an orientable flat manifold in dimension 4 with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as holonomy group. As already stressed before we only have to consider the orientable manifolds. Then we may assume that the fundamental group $\pi_1(M)$ is generated by $(e_1, I_4), (e_2, I_4), (e_3, I_3), (e_4, I_4), (a_1, A_1)$ and (a_2, A_2), with 1 in the i-th place for e_i and 0 everywhere else; $A_1, A_2 \in \mathrm{Gl}(4, \mathbb{Z})$ commuting matrices of order 2 and a_1, a_2 suitable translational parts. Since A_1 and A_2 commute we know that there exist a $P \in \mathrm{Gl}(n, \mathbb{Q})$ such that

$$PA_1P^{-1} = B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } PA_2P^{-1} = B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The multiplicity of -1 must indeed be 2 since M needs to be orientable (see also Proposition 1.6).

As explained before, in order to verify the validity of the Anosov theorem for a specific manifold, we need a description of all suitable affine endomorphisms $(\delta, \mathfrak{D}) \in \mathbb{R}^4 \rtimes \mathrm{Endo}(\mathbb{R}^4)$. To construct all these endomorphisms we actually have to consider all the possible homomorphisms $\theta : \pi_1(M) \to \pi_1(M)$, since for each $(a, A) \in \pi_1(M)$:

$$\theta((a, A))(\delta, \mathfrak{D}) = (\delta, \mathfrak{D})(a, A)$$

Suppose that $\theta((a,A)) = (b,B)$, then this boils down to

$$(b + B\delta, B\mathfrak{D}) = (\delta + \mathfrak{D}a, \mathfrak{D}A)$$

This equation gives us two types of conditions which need to be satisfied, namely one type concerning the linear parts and another concerning the translational parts. Therefore in a first step we construct all possible matrices $\mathfrak{D}$ such that $B\mathfrak{D} = \mathfrak{D}A$. In a second step we then look at the consequences of the fact that $\delta + \mathfrak{D}a = b + B\delta$. This second step has to be done separately for each of the 9 manifolds under study, but the work of the first step can be simplified a lot because of the existence of P. Indeed we only have to search for all matrices $\mathfrak{D}$ such that $PBP^{-1}\mathfrak{D} = \mathfrak{D}PAP^{-1}$ since

$$PBP^{-1}\mathfrak{D} = \mathfrak{D}PAP^{-1} \Leftrightarrow BP^{-1}\mathfrak{D}P = P^{-1}\mathfrak{D}PA$$

So $P^{-1}\mathfrak{D}P$ gives us then a full description of all possible matrices. The list below contains 16 possible matrices, since in order to construct all possible homomorphisms θ, we consider the 16 possible conditions $X\mathfrak{D} = \mathfrak{D}B_i$, with $i = 1, 2$ and $X \in \{I_4,\, B_1,\, B_2,\, B_1B_2\}$.

We could (and perhaps should) in fact further partition each of these sixteen cases into 4 more cases, depending upon the four possibilities $X\mathfrak{D} = \mathfrak{D}I_4$, for $X = I_1,\ B_1,\ B_2$ or B_1B_1. This corresponds to the fact that the image of a translation under the homomorphism θ does not need to be a translation. These extra conditions will of course not lead to 'new possibilities' for $\mathfrak{D}$, but they will only imply that extra zeroes will appear in the possible $\mathfrak{D}$'s.

The 16 matrices are the following. Note that the d_{ij} will be specified later on.

$\mathfrak{D}_1 = \mathfrak{D}_1 B_1$ $\mathfrak{D}_1 = \mathfrak{D}_1 B_2$ $\mathfrak{D}_1 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 0 & 0 & 0 \\ d_{31} & 0 & 0 & 0 \\ d_{41} & 0 & 0 & 0 \end{pmatrix}$	$\mathfrak{D}_2 = \mathfrak{D}_2 B_1$ $B_1\mathfrak{D}_2 = \mathfrak{D}_2 B_2$ $\mathfrak{D}_2 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 0 & 0 & 0 \\ 0 & d_{32} & 0 & 0 \\ 0 & d_{42} & 0 & 0 \end{pmatrix}$
$\mathfrak{D}_3 = \mathfrak{D}_3 B_1$ $B_2\mathfrak{D}_3 = \mathfrak{D}_3 B_2$ $\mathfrak{D}_3 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & d_{32} & 0 & 0 \\ d_{41} & 0 & 0 & 0 \end{pmatrix}$	$\mathfrak{D}_4 = \mathfrak{D}_4 B_1$ $B_1 B_2\mathfrak{D}_4 = \mathfrak{D}_4 B_2$ $\mathfrak{D}_4 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & 0 & 0 & 0 \\ 0 & d_{42} & 0 & 0 \end{pmatrix}$
$B_1\mathfrak{D}_5 = \mathfrak{D}_5 B_1$ $\mathfrak{D}_5 = \mathfrak{D}_5 B_2$ $\mathfrak{D}_5 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{34} \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	$B_1\mathfrak{D}_6 = \mathfrak{D}_6 B_1$ $B_1\mathfrak{D}_6 = \mathfrak{D}_6 B_2$ $\mathfrak{D}_6 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ d_{21} & 0 & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & d_{43} & 0 \end{pmatrix}$
$B_1\mathfrak{D}_7 = \mathfrak{D}_7 B_1$ $B_2\mathfrak{D}_7 = \mathfrak{D}_7 B_2$ $\mathfrak{D}_7 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	$B_1\mathfrak{D}_8 = \mathfrak{D}_8 B_1$ $B_1 B_2\mathfrak{D}_8 = \mathfrak{D}_8 B_2$ $\mathfrak{D}_8 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & 0 & d_{34} \\ 0 & 0 & d_{43} & 0 \end{pmatrix}$

$B_2\mathfrak{D}_9 = \mathfrak{D}_9 B_1$ $\mathfrak{D}_9 = \mathfrak{D}_9 B_2$ $\mathfrak{D}_9 = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{24} \\ 0 & 0 & 0 & d_{34} \\ d_{41} & 0 & 0 & 0 \end{pmatrix}$	$B_2\mathfrak{D}_{10} = \mathfrak{D}_{10} B_1$ $B_1\mathfrak{D}_{10} = \mathfrak{D}_{10} B_2$ $\mathfrak{D}_{10} = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{24} \\ 0 & 0 & d_{33} & 0 \\ 0 & d_{42} & 0 & 0 \end{pmatrix}$
$B_2\mathfrak{D}_{11} = \mathfrak{D}_{11} B_1$ $B_2\mathfrak{D}_{11} = \mathfrak{D}_{11} B_2$ $\mathfrak{D}_{11} = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & d_{23} & 0 \\ 0 & 0 & d_{33} & 0 \\ d_{41} & 0 & 0 & 0 \end{pmatrix}$	$B_2\mathfrak{D}_{12} = \mathfrak{D}_{12} B_1$ $B_1B_2\mathfrak{D}_{12} = \mathfrak{D}_{12} B_2$ $\mathfrak{D}_{12} = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & d_{23} & 0 \\ 0 & 0 & 0 & d_{34} \\ 0 & d_{42} & 0 & 0 \end{pmatrix}$
$B_1B_2\mathfrak{D}_{13} = \mathfrak{D}_{13} B_1$ $\mathfrak{D}_{13} = \mathfrak{D}_{13} B_2$ $\mathfrak{D}_{13} = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{24} \\ d_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	$B_1B_2\mathfrak{D}_{14} = \mathfrak{D}_{14} B_1$ $B_1\mathfrak{D}_{14} = \mathfrak{D}_{14} B_2$ $\mathfrak{D}_{14} = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{24} \\ 0 & d_{32} & 0 & 0 \\ 0 & 0 & d_{43} & 0 \end{pmatrix}$
$B_1B_2\mathfrak{D}_{15} = \mathfrak{D}_{15} B_1$ $B_2\mathfrak{D}_{15} = \mathfrak{D}_{15} B_2$ $\mathfrak{D}_{15} = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & d_{23} & 0 \\ 0 & d_{32} & 0 & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	$B_1B_2\mathfrak{D}_{16} = \mathfrak{D}_{16} B_1$ $B_1B_2\mathfrak{D}_{16} = \mathfrak{D}_{16} B_2$ $\mathfrak{D}_{16} = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & 0 & d_{23} & 0 \\ d_{31} & 0 & 0 & 0 \\ 0 & 0 & d_{43} & 0 \end{pmatrix}$

Remark 7.1. *Note that this method and matrices can be adapted to cover any orientable flat manifold with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. For such manifolds the $\mathbb{Q}$-irreducible components are one-dimensional but their multiplicity needs no longer to be* 1 *(as is the case for the manifolds above). Using the techniques described above one again obtains* 16 *possible linear parts but now the d_{ij} have to be replaced by matrices*

D_{ij} of the 'right' dimensions. This comes from the fact that there are $\mathbb{Q}$-irreducible components with multiplicity greater then one.

So there are 16 cases which we need to examine, but for 6 cases we can already state that the Anosov relation holds. Indeed, if we calculate the determinants in Theorem 2.5 for $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_9$ and $\mathfrak{D}_{16}$ we always obtain $1 - d_{11}$. So whatever the value of d_{11} is, we always have that the Anosov relation holds. Secondly for the cases $\mathfrak{D}_{12}$ and $\mathfrak{D}_{14}$ the conditions in Proposition 2.6 are satisfied which again implies that the Anosov relation holds. So, in the discussion below we no longer have to include these linear parts.

Evaluating the second type of conditions on the translational parts leads to some interesting results for the remaining linear parts. This comes from the fact that, as is for instance also done in Chapter 6, in this step we specify the entries of the matrices mentioned above. Moreover in some cases we have to exclude many of the matrices since the entries of the matrices can not satisfy the conditions coming from the translational parts. This is nicely demonstrated for this first manifold.

Let E_1 be the first Bieberbach group in 22.1.1 which is generated by

$$(e_i, I_4), \qquad (a_1, A_1) = (\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix})$$

$$\text{and } (a_2, A_2) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix})$$

with 1 in the i-th place and 0 everywhere else for e_i; $i = 1, \dots 4$. To be precise we slightly transformed the matrices found on [7] to avoid the explicit use of the matrix P. Then $M_1 = E_1 \backslash \mathbb{R}^4$ is indeed one of the manifolds under study.

Hereafter, we only focus on the linear parts $\mathfrak{D}$, and not on the translational parts δ, since the validity of the Anosov theorem only depends on $\mathfrak{D}$. For this manifold we have that from the 10 remaining possibilities only $\mathfrak{D}_7, \mathfrak{D}_8, \mathfrak{D}_{10}, \mathfrak{D}_{12}, \mathfrak{D}_{14}$ and $\mathfrak{D}_{15}$ are suitable linear parts under the extra condition that each non-zero entry is an odd integer except for d_{11} which can be any integer. Note that possibly the same phenomenon can happen for the 6 cases that we do not consider.

Let us demonstrate the calculations that need to be done for instance for $\mathfrak{D}_7$. In that case we actually assume that $\theta(a_i, A_i) = (z_i + a_i, A_i)$ with $z_i \in \mathbb{Z}_4$ and $i = 1, 2$. So from the second type of conditions on the translational parts, we obtain that $z_i + a_i + A_i\delta = \delta + \mathfrak{D}_7 a_i$ or equivalently that

$$(I_4 - A_i)\delta + (\mathfrak{D}_7 - I_4)a_i \in \mathbb{Z}^4.$$

With $\delta = (t_1, t_2, t_3, t_4)^t$ this implies for $i = 1$ that

$$\begin{pmatrix} 0 \\ \frac{1}{2}(d_{22} - 1) \\ 2t_3 + \frac{1}{2}(d_{33} - 1) \\ 2t_4 \end{pmatrix} \in \mathbb{Z}_4$$

and for $i = 2$ that

$$\begin{pmatrix} 0 \\ 2t_2 \\ 2t_3 \\ (d_{44} - 1)\frac{1}{2} \end{pmatrix} \in \mathbb{Z}_4.$$

So d_{22} and d_{44} must be odd integers and since $2t_3$ is an integer the same must hold for d_{33}. To show that d_{11} must be an integer we have to look at $\theta(e_1, I_4)$. Suppose that $\theta(e_1, I_4) = (b, B)$, then we know from the fact that the first entry of the translational parts a_1 and a_2 is zero that the first entry of b must be an integer. Analogously calculations as the one above, imply then that d_{11} also must be an integer.

The 4 other possible linear parts are excluded since we always obtain that some zero entry must also be an odd integer. Let us again demonstrate this, for instance for $\mathfrak{D}_3$. In that case we assume that $\theta(a_2, A_2) = (z_2 + a_2, A_2)$ with $z_2 \in \mathbb{Z}_4$. In the same way as above this implies that

$$\begin{pmatrix} 0 \\ 2t_2 \\ 2t_3 \\ \frac{-1}{2} \end{pmatrix} \in \mathbb{Z}_4.$$

which clearly gives us a contradiction.

Under the given conditions for the possible linear parts one can verify, using for instance the Lemmas 6.8, 6.9 and of course Theorem 2.5 that the Anosov theorem holds for M_1.

Let us now look at the second Bieberbach group E_2 of 22.1.1 by using the above representation but now we replace a_1 by $\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Note that $M_2 = E_2 \backslash \mathbb{R}^4$ is in the same $\mathbb{Z}$-class as M_1. However, in opposite to the results for M_1, we are able to construct a map on M_2 which does not satisfy the Anosov relation. Namely let $f_2 : M_2 \to M_2$ be induced by

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}).$$

As before one can verify that this affine endomorphism indeed induces a map on M_2. The explanation for this example is quite simple, because for M_2 we no longer have the condition that d_{22} has to be an odd integer. This comes from the fact that in this case the second entry of a_1 is now also equal to 0. Therefore we can take d_{22} equal to zero and so we obtain that $\det(I_4 - \mathfrak{D}) = -8$ and $\det(I_4 - A_1\mathfrak{D}) = 16$ which by Theorem 2.5 implies that the Anosov relation does not hold for f_2.

A similar example works for the third orientable manifold M_3 obtained by using the third Bieberbach group of 22.1.1. Namely let $f_3 : M_3 \to M_3$ be induced by

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}).$$

Another interesting example is the fourth Bieberbach group of 22.1.1. Replacing in the above representation a_1 by $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$ gives us a Bieberbach group E_4 and another manifold $M_4 = E_4 \backslash \mathbb{R}^4$ of the same $\mathbb{Z}$-class. For this manifold we again have that $\mathfrak{D}_7, \mathfrak{D}_8, \mathfrak{D}_{10}, \mathfrak{D}_{12}, \mathfrak{D}_{14}$ and $\mathfrak{D}_{15}$ are suitable linear parts but by the same reasons as for the first manifold we can not use them to construct a counterexample for the Anosov relation.

Now however $\mathfrak{D}_5$ and $\mathfrak{D}_{13}$ are also suitable linear parts under the restrictions that the entries of the first column are odd integers and the entries of the last column are even integers. Let $f_4 : M_4 \to M_4$ be induced by

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} 3\,0\,0\,0 \\ 3\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,2 \end{pmatrix}),$$

then the Anosov relation does not hold for f_4 since $\det(I_4 - \mathfrak{D}) = 2$ and $\det(I_4 - A_1\mathfrak{D}) = -6$. Note that it is impossible to find a counter example in which the δ-part is equal to zero.

This example shows that to construct counter examples we can not always use affine endomorphisms with a linear part of the form $\mathfrak{D}_7$. To be precise, from Theorem 1.10 we know that for every affine endomorphism $(\delta, \mathfrak{D})$ there exists a unique morphism θ on the universal covering group. In every counterexample up till now, we always used a θ which maps every element of the fundamental group to itself modulo a translation. Amongst other things this implies for every element (a, A) of the fundamental group that $A\mathfrak{D} = \mathfrak{D}A$. Or equivalently that the associated matrices commute and this simplifies a lot the calculations that need to be done. So a possible question could be: if there exist a counterexample, can one always find then a counterexample such that the matrices of the linear parts of the elements of the fundamental group commute with the linear part of the affine endomorphism? M_4 answers this question negatively.

For the other 5 manifolds of this class similar calculations show that the Anosov theorem holds for the 3 manifolds corresponding to the Bieberbach groups of respectively 22.1.4, 22.1.8 and 22.1.12. The counterexamples for the 2 other manifolds are presented below. The Anosov relation does not hold for the map f_5 on the manifold corresponding to the Bieberbach group of 22.1.2 and induced by

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3\,0\,0\,0 \\ 0\,3\,0\,0 \\ 0\,0\,2\,2 \\ 0\,0\,2\,2 \end{pmatrix}).$$

Similarly for the map f_6 on the manifold corresponding to the Bieberbach group of 22.1.5 and induced by

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \end{pmatrix}).$$

To conclude we may state that this class of manifolds is very interesting and gives us very nice counterexamples which already negatively answers some research questions. Such as for instance: can you always make a counterexample such that the linear part of the homotopy lift commutes with the linear parts of the generators of the fundamental group?
Secondly, it also shows that the translational parts of the elements of the fundamental group can be very crucial for the validity of the Anosov theorem. Except for the results of Chapter 6 this was never the case for the results of the previous chapters.
Finally we proved the following rather surprising result concerning $\mathbb{Z}$-classes of Bieberbach groups (= fundamental groups of flat manifolds).

Proposition 7.2. *There exists flat manifolds M_1 and M_2 such that their fundamental groups belong to the same $\mathbb{Z}$-class and such that the Anosov theorem holds for M_1 but not for M_2.*

7.3 Flat manifolds in dimension 4 with non-abelian holonomy group

While in the previous section the flat manifolds have an abelian holonomy group, this no longer holds for this section. This is interesting since most of our theorems are results concerning manifolds with abelian holonomy group (see chapters 5 and 6). So a next step for future research could be trying to find some extra classes of infra-nilmanifolds with non-abelian holonomy group for which the Anosov theorem holds. In dimension 4 there are 14 of these manifolds, from which there are 10 orientable. The holonomy groups of these 10 manifolds are $S_3, \mathbb{Z}_2 \oplus S_3, D_8$ and A_4.

Moreover, dimension 4 is the lowest dimension in which there exist manifolds having one of the above groups as holonomy group. Now, one could be interested in the fact wether the Anosov theorem holds for infra-nilmanifolds, having a specific holonomy group and appearing in the lowest possible dimension. Or under which extra conditions this could hold. The Klein bottle is a 'counterexample' since dimension 2 is the minimal dimension in which $\mathbb{Z}_2$ occurs as a holonomy group, but nevertheless the Anosov theorem does not hold. On the other hand orientability is for generalized Hantzsche-Wendt manifolds an example of such an extra condition.

For the three flat manifolds with holonomy group S_3 and the one flat manifold with holonomy group $\mathbb{Z}_2 \oplus S_3$ the Anosov theorem does not hold. Since for each of these manifolds we can easily construct a counterexample. For instance let E_1 be the Bieberbach group of 38.1.1 generated by

$$(e_i, I_4), \qquad (a_1, A_1) = (\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix})$$

$$\text{and } (a_2, A_2) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix})$$

with 1 in the i-th place and 0 everywhere else for e_i; $i = 1, \ldots 4$.
Then $M_1 = E_1 \backslash \mathbb{R}^4$ is indeed a flat orientable manifold with S_3 as holonomy group. Let $f_1 : M_1 \to M_1$ be induced by

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}).$$

Then $(\delta, \mathfrak{D})$ indeed induces a map on M_1 and $\det(I_4 - \mathfrak{D}) = 6$ and $\det(I_4 - A_2\mathfrak{D}) = -10$. The same example works if we replace E_1 by the Bieberbach group of 38.2.1.
Analogous results holds for the Bieberbach group of 38.2.3 with

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix}$$

and the Bieberbach group of 81.1.1 with

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}).$$

On the other hand for the two flat manifolds with A_4 as holonomy group the Anosov theorem holds. To prove this statement we need some notations. Let E_2 be the Bieberbach group of 44.2.1 generated by

$$(e_i, I_4), \qquad (b_1, B_1) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix})$$

$$\text{and } (b_2, B_2) = (\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix})$$

and E_3 the Bieberbach group of 44.1.2 generated by

$$(e_i, I_4), \qquad (c_1, C_1) = (\begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix})$$

$$\text{and } (c_2, C_2) = (\begin{pmatrix} \frac{5}{6} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix})$$

with 1 in the i-th place and 0 everywhere else for e_i; $i = 1, \dots 4$.
Then $M_2 = E_2 \backslash \mathbb{R}^4$ and $M_3 = E_3 \backslash \mathbb{R}^4$ are the manifolds under study. To verify our statement we work very similar to the previous section. Namely we start with calculating all possible linear parts for M_2 but now we have to consider 12 possibilities for as well B_1 as B_2 which result in 144 possible matrices. However 120 of these matrices are of the form

$$\begin{pmatrix} 0 & 0 & 0 & d_{14} \\ 0 & 0 & 0 & d_{24} \\ 0 & 0 & 0 & d_{34} \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$$

with $d_{ij} \in \mathbb{Z}$. For instance, consider the case where $\mathfrak{D}B_1 = \mathfrak{D}$ and $B_1\mathfrak{D} = \mathfrak{D}B_2$, then

$$\mathfrak{D} = \begin{pmatrix} 0 & 0 & 0 & d_{14} \\ 0 & 0 & 0 & -d_{14} \\ 0 & 0 & 0 & d_{14} \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$$

Note that this example also shows that there could be some extra restrictions on the d_{j4}. Using Theorem 2.5 we easily obtain that maps induced on M_2 by affine endomorphisms with a linear part of the above form always satisfy the Anosov relation. Note that the same holds for M_3 since we have that $C_1 = PB_1P^{-1}$ and $C_2 = PB_2P^{-1}$ with

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

So again we only have to calculate the possible linear parts once. The other 24 matrices are of the form $\begin{pmatrix} D & 0 \\ 0 & d_{44} \end{pmatrix}$ with D a (3×3)-matrix with in each row and each column exactly one non-zero element. For instance, consider the case where $B_1^2B_2\mathfrak{D} = \mathfrak{D}B_2$ and $B_2\mathfrak{D} = \mathfrak{D}B_2$, then

$$\mathfrak{D} = \begin{pmatrix} 0 & 0 & d_{13} & 0 \\ 0 & d_{13} & 0 & 0 \\ -d_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}.$$

Moreover, what is demonstrated in this example happens in any case. Namely, suppose that in the first row k appears, then in the other rows only $\pm k$ can appear. For instance D can be

$$\begin{pmatrix} 0 & k & 0 \\ 0 & 0 & -k \\ k & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & k \end{pmatrix}.$$

If we then calculate the determinants of Theorem 2.5 for each of these possible linear parts, we always obtain that the 12 determinants are of the form

$$(1 - d_{44})(1 - k)X \tag{7.1}$$

with $X \geq 0$. For instance if we use our example once more we find two possible values for the determinants: $(1 - d_{44})(1 - d_{13})(1 + d_{13}^2)$ and $(1 - d_{44})(1 - d_{13})(1 + d_{13})^2$.

The equation in 7.1 implies that the Anosov relation must hold for the maps on M_2 induced by affine endomorphisms with such a linear part since clearly all these determinants have the same sign.

Again, because of P the same holds for M_3 since, as is demonstrated before, P has 'no influence' on the sign of the determinants in Theorem 2.5.

It is important to note that we do not have to analyze the second type of conditions which come from the translational parts since we already know that the Anosov relation holds. However analyzing these conditions can, as is demonstrated in the previous section, possibly reduce the number of suitable linear parts. We did not verify this since we already had an answer to our question.

Finally for the 4 flat manifolds with D_8 as holonomy group the situation is very similar to the case of the 4-dimensional flat manifolds with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as holonomy group. Namely, for the infra-nilmanifolds obtained from the first and the third Bieberbach group of 29.1.1 we can construct a counterexample. These counterexamples are respectively induced by

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&5&0 \\ 0&0&0&5 \end{pmatrix})$$

and by

$$(\delta, \mathfrak{D}) = (\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{8} \end{pmatrix}, \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 0&0&3&0 \\ 0&0&0&4 \end{pmatrix}).$$

Note that for the second counterexample, the same holds as in the case of the fourth Bieberbach group of 22.1.1. Namely again, we are not able to use an affine endomorphism for which the linear part commutes with the linear parts of the generators of the fundamental group.

For the manifolds corresponding to the second Bieberbach group of 29.1.1 and the Bieberbach group of 29.1.2 we found by similar calculations as before that the Anosov theorem does hold. So again we found 2 manifolds from the same $\mathbb{Z}$-class for which the Anosov theorem only holds in one case. Therefore the observations of the previous sections are confirmed by these manifolds.

To conclude we may state that these 10 manifolds were very interesting to investigate. They showed for instance that the minimal dimension need not to be restrictive (see S_3), that similar things may happen as in the case of abelian holonomy groups (see D_8) and most interestingly we perhaps found the top of the iceberg of a new class of manifolds for which the Anosov theorem holds (see A_4). Which leads to the following question.

Question 7.3. *Does the Anosov theorem hold for any (flat) manifold with holonomy group A_4?*

Chapter 8

Infra-nilmanifolds

In principle, for infra-nilmanifolds we have to do the same calculations as for flat manifolds. Namely, again by the methods described before, we have to calculate all possible affine endomorphisms and verify Theorem 2.5. However there are two complications. First of all the known descriptions of the almost-Bieberbach groups in low dimensions are not as 'nice' as in the flat case. E.g. from the given group presentation in [12] one can not readily obtain the needed translational and linear parts of a given element. Secondly, the fact that we are now working with non-abelian Lie groups, implies that there are extra restrictions on the linear parts of the suitable affine endomorphisms.

Therefore we start this chapter by formally showing how the calculations can be carried out for 4-dimensional 2-step nilpotent infra-nilmanifolds. The 3-dimensional case is completely similar but less complicated. We also briefly indicate how we deal with the 4-dimensional, 3-step nilpotent case.

In the following 3 sections we then summarize the results and as before we present all the counter examples and give the proofs of our statements.

As in the previous chapter, we need to be able to refer explicitly to specific infra-nilmanifolds. Since an infra-nilmanifold is also completely determined by its fundamental group, we now need a classification of the almost-Bieberbach groups. Such a classification is available in [12], where the almost-Bieberbach groups are divided into numbered families, each still depending on parameters k_i. In dimension 3 there are at most 4 of these parameters, while in dimension 4 there are at most 7. For each family, the possible parameters are specified and a presentation of each almost-Bieberbach group belonging to this family is given. Besides this presentation, also a matrix representation λ of the almost-Bieberbach group is given. For example, for the second family in dimension 3 we find the following entry ([12, p.160]).

$$\begin{aligned} E = \langle a, b, c, \alpha \mid {} & [b,a] = c^{k_1}, [c,a] = 1, [c,b] = 1, \\ & \alpha c = c\alpha, \alpha a = a^{-1}\alpha c^{k_2}, \alpha b = b^{-1}\alpha c^{k_3}, \\ & \alpha^2 = c^{k_4} \rangle \end{aligned}$$

$$\lambda(\alpha) = \begin{pmatrix} 1 & k_2 & k_3 & \frac{k_4}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{AB-groups:} \quad k > 0, k \equiv 0 \bmod 2, E = \langle (k, 0, 0, 1) \rangle$$

For any possible choice of an integral value of the parameters k_i, the presentation given above, determines an almost-crystallographic group, generated by four elements, a, b, c and α. There is no full information on possible isomorphisms between almost-crystallographic groups determined by different choices of the parameters. However, full information is available for the subset of almost-Bieberbach groups, the torsion free almost-crystallographic groups.

Indeed, the possible almost-Bieberbach groups E are given by the condition on the parameters denoted in the last line. In the case above, this states that $k_1 = k$ must be a positive and even integer, that $k_2 = k_3 = 0$ and that $k_4 = 1$. Note that other choices of parameters can also lead to an almost-Bieberbach group, but then this group will be isomorphic to exactly one with a set of parameters as listed in the last line of the data above.

For any of the almost-crystallographic groups determined by a presentation as above, the a, b, c always generate the Fitting subgroup N (so the lattice in the corresponding Lie group) and the restriction of the matrix representation to this Fitting subgroup can be described in a general form for all the almost-crystallographic groups in dimension 3 at once. Namely, in all cases we have that $[b, a] = c^l, [c, a] = 1, [c, b] = 1$ for some integer l and then the matrix representation λ is given by

$$\lambda(a) = \begin{pmatrix} 1 & 0 & \frac{-l}{2} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \lambda(b) = \begin{pmatrix} 1 & \frac{l}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \lambda(c) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Note that in our example above $l = k_1$). This representation of the Fitting subgroup extends to a representation of the full almost-crystallographic group by defining the images of generators as indicated in the entry above. We will explain in the next section, how this affine representation really contains all the information we need.

In the 4-dimensional case analogous entries can be found, but they are slightly more complicated since for instance now four generators of the Fitting subgroup are given and there are more relations between them. Moreover, we also have to consider 3-step nilpotent Lie groups.

Finally, note that during this chapter, G always denotes the Lie group corresponding to the almost-Bieberbach group E under study (or the universal covering group of the infra-nilmanifold) and N refers to the Fitting subgroup in E (which is also the corresponding lattice in G).

8.1 Calculations on 4 dimensional infra-nilmanifolds

In this section we shortly explain how we are able to make the necessary calculations with the almost Bieberbach groups we need to investigate. We first present a construction which can be used in general for the 2-step nilpotent case, but not for higher nilpotency classes. Afterwards we give, in a second section, a specific method, which we can use in the 4-dimensional 3-step nilpotent case.

8.1.1 2-step nilpotent infra-nilmanifolds

In this section we demonstrate how we can use the affine representation of the fundamental group to obtain the information that we need. We do this in the (more general) case where the universal covering group is a 4-dimensional, 2-step nilpotent Lie group. Therefore, let $\mathfrak{g}$ be the 4-dimensional, 2-step nilpotent Lie algebra determined by

$$\begin{aligned} \mathfrak{g} = \langle A, B, C, D \mid & [B, A] = l_1 D, [C, A] = l_2 D, [C, B] = l_3 D, \\ & [D, A] = 0, [D, B] = 0, [D, C] = 0 \rangle \end{aligned}$$

Here, $l_1, l_2, l_3 \in \mathbb{Z}$. (In fact, any non-zero choice of parameters actually determines, up to some isomorphism, the same Lie algebra, but this is not important to what follows).
Note that in this chapter we use the notations of [12] and we refer to this book for more details. This algebra has a faithful matrix representation:

$$\varphi : \mathfrak{g} \to M_5(\mathbb{R}) : xA+yB+zC+tD \mapsto \begin{pmatrix} 0 & \frac{l_1y+l_2z}{2} & \frac{-l_1x+l_3z}{2} & \frac{-l_1x-l_2y}{2} & t \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $G = \exp(\mathfrak{g})$ is a connected, simply connected, 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$ and a faithful matrix representation is given by

$$\psi : G \to \mathrm{Gl}(5,\mathbb{R}) : \exp(xA+yB+zC+tD) \mapsto \exp(\varphi(xA+yB+zC+tD))$$

Let $a = \exp(A), b = \exp(B), c = \exp(C), d = \exp(D)$, and let N be the subgroup of G, generated by a, b, c and d. One can check that N has a presentation:

$$\begin{aligned} N = \langle a, b, c, d \mid [b,a] = d^{l_1}, [c,a] = d^{l_2}, [c,b] = d^{l_3}, \\ [d,a] = 1, [d,b] = 1, [d,c] = 1\rangle. \end{aligned}$$

Later on, we need to able to work with z^r for any $z \in N, r \in \mathbb{R}$, which is obtained as follows $z^r = \exp(r \log z)$.

In [12, p 168], we find a matrix representation $\lambda : N \to \mathrm{Gl}(5,\mathbb{R})$ for N, given by

$$\lambda(a) = \begin{pmatrix} 1 & 0 & \frac{-l_1}{2} & \frac{-l_2}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \lambda(b) = \begin{pmatrix} 1 & \frac{l_1}{2} & 0 & \frac{-l_3}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\lambda(c) = \begin{pmatrix} 1 & \frac{l_2}{2} & \frac{l_3}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \lambda(d) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is obvious that λ is just the restriction of ψ to N. Moreover, N is a lattice of G (or $\lambda(N)$ is a lattice of $\psi(G)$) and thus G is the Mal'cev completion of N.

Now we want to extend the above matrix representation ψ, to obtain a matrix representation of $G \rtimes \mathrm{Endo}(G)$. Therefore take $\mathfrak{D} \in \mathrm{Endo}(G)$, then $\mathfrak{D}_*$ is also an element of Endo($\mathfrak{g}$) and $\mathfrak{D}_*$ is described by a matrix of the form

$$T_{\mathfrak{D}_*} = \begin{pmatrix} \epsilon & p_1 & q_1 & r_1 \\ 0 & p_2 & q_2 & r_2 \\ 0 & p_3 & q_3 & r_3 \\ 0 & p_4 & q_4 & r_4 \end{pmatrix} \tag{8.1}$$

with respect to the basis D, A, B, C. Moreover ϵ must satisfy certain, depending on the values of the parameters l_i, of the following relations:

$$\begin{aligned}
\epsilon l_1 &= l_1(p_2q_3 - q_2p_3) + l_2(p_2q_4 - q_2p_4) + l_3(p_3q_4 - q_3p_4) \\
\epsilon l_2 &= l_1(p_2r_3 - r_2p_3) + l_2(p_2r_4 - r_2p_4) + l_3(p_3r_4 - r_3p4) \\
\epsilon l_3 &= l_1(q_2r_3 - r_2q_3) + l_2(q_2r_4 - r_2q_4) + l_3(q_3r_4 - r_3q_4)
\end{aligned}$$

The i^{th} equation has to be satisfied if and only if $l_i \neq 0$.
Note that also the converse holds: any matrix of the above form is the matrix of the differential of an element of Endo(G). An easy calculation, using (8.1), then shows that

$$\chi : G \rtimes \mathrm{Endo}(G) \to M_5(\mathbb{R}) : (\delta, \mathfrak{D}) \mapsto \psi(\delta) \begin{pmatrix} T_{\mathfrak{D}_*} & 0 \\ 0 & 1 \end{pmatrix}$$

is a faithful representation of the semi-group $G \rtimes \mathrm{Endo}(G)$ and note that $\chi(G \rtimes \mathrm{Aut}(G)) = \chi(G \rtimes \mathrm{Endo}(G)) \cap \mathrm{Gl}(5, \mathbb{R})$. Obviously χ extends ψ and λ.

If we now consider any 4-dimensional, 2-step nilpotent Bieberbach group E as listed in [12], then it is clear that $\lambda(E) \subseteq \chi(G \rtimes \mathrm{Aut}(G))$. This implies that, using the faithful representations λ and χ, we can view E as a discrete subgroup of $G \rtimes \mathrm{Aut}(G)$ such that $G \cap E = N$ and E/N is finite. Using this construction and the matrices of [12], we can now easily make calculations in $G \rtimes \mathrm{Endo}(G)$. To do that, one has to decompose the matrices of the generators into a product of the matrix associated to the translational part and the matrix associated to the linear part. Once we have these matrices, we can proceed analogously to the calculations for the flat manifolds. Let us illustrate this by means of an example using the third family in dimension 4 ([12, p 170].

Example 8.1. *In [12] we find the following entry.*

$$\begin{aligned} E = \langle a, b, c, d\alpha \mid\ & [b,a] = 1, [c,a] = d^{k_1}, [c,b] = 1, \\ & \alpha c = c^{-1}\alpha d^{k_3}, \alpha a = a^{-1}\alpha d^{k_2}, \alpha b = b\alpha, \\ & \alpha d = d\alpha, \alpha^2 = d^{k_4} \rangle \end{aligned}$$

$$\lambda(\alpha) = \begin{pmatrix} 1 & k_2 & 0 & k_3 & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

AB-groups: $\quad k > 0, k \equiv 0 \bmod 2, E = \langle (k, 0, 0, 1) \rangle$

This implies that for the almost-Bieberbach groups E of this family that $l_1 = l_3 = k_2 = k_3 = 0$, $k_4 = 1$, l_2 is an even integer and

$$\lambda(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \lambda(d^{\frac{1}{2}}) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By the method described above, this implies that the translational part of α is equal to $d^{\frac{1}{2}}$ and that the differential of the linear part is described by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

From this we may conclude that the infra-nilmanifold which has E as its fundamental group is orientable and has $\mathbb{Z}_2$ as holonomy group.
Let $(\delta, \mathfrak{D})$ be an affine endomorphism of G, then $\mathfrak{D}_*$ is represented by a matrix of the form

$$\begin{pmatrix} \epsilon & p_1 & q_1 & r_1 \\ 0 & p_2 & q_2 & r_2 \\ 0 & p_3 & q_3 & r_3 \\ 0 & p_4 & q_4 & r_4 \end{pmatrix}$$

satisfying $\epsilon l_2 = l_2(p_2 r_4 - r_2 p4) \Leftrightarrow \epsilon = p_2 r_4 - r_2 p_4$.

An analogous, but easier construction, can be carried out in the three dimensional case.

8.1.2 3-step nilpotent infra-nilmanifolds

In this section we briefly describe the method we used for the 3-step nilpotent case. Since we restrict ourselves to dimension 4, we find in [12] that we only have to deal with infra-nilmanifolds with holonomy group $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover in the latter case the manifolds are always non-orientable and therefore we only focus on the infra-nilmanifolds with $\mathbb{Z}_2$ as holonomy group.

Assume that E is a fundamental group of such an infra-nilmanifold and is generated by a, b, c, d, α. Again we want to replace α by an affine map of the form $(a^{t_1}b^{t_2}c^{t_3}d^{t_4}, \mathfrak{A})$. (Note that we are still assuming that G is the Mal'cev completion of the lattice N generated by a, b, c, d and that $\mathfrak{g}$ is the corresponding Lie algebra generated by D, C, A, B.) In the previous section we could avoid the actual construction of $\mathfrak{A}$ by using only $\mathfrak{A}_*$. Now, we will really need $\mathfrak{A}$ in order to construct $\mathfrak{A}_*$. However, the first thing we will do is to determine the translational part of the generator α with a non-trivial linear part. For this, we explicitly use that the order of α is 2. This implies that we can find $t_1, \ldots, t_4$ by using the respective matrix representations of a, b, c, d, α and the fact that

$$\lambda(((a^{t_1}b^{t_2}c^{t_3}d^{t_4})^{-1}\alpha)^2) = I_5$$

Note that in the 3-step nilpotent case the matrix representation of the lattice is not the same for all the almost-Bieberbach groups. In the following we will denote $(a^{t_1}b^{t_2}c^{t_3}d^{t_4})^{-1}\alpha$, with the t_i as found above, by β.

We can now determine $\mathfrak{A}_*$ by explicitly computing the images of D, C, A, B. The image of for instance A is given by

$$\begin{aligned} \mathfrak{A}_*(A) &= \log(\mathfrak{A}(\exp(a))) \\ &= q_1 D + q_2 C q_3 A +_4 B \end{aligned}$$

with $q_i \in \mathbb{R}$ and analogously for the other elements. Now to calculate this we can use the matrix representations, since we have that $\mathfrak{A}(g) = \lambda(\beta g \beta^{-1}) = \lambda(\beta g \beta)$. This allows us to explicitly determine $\mathfrak{A}_*$. Note that because of the structure of G, we have that $\mathfrak{A}_*(D)$ only depends on D and $\mathfrak{A}_*(C)$ only on D, C. To calculate the respective coefficients we need the relations between D, C, A, B and the images of A and B.

Of course the same can be done to calculate the δ and $\mathfrak{D}_*$ of the suitable affine endomorphisms $(\delta, \mathfrak{D})$. Note that since we may assume that $\mathbb{Z}_2$ is the holonomy group, we may because of Proposition 2.6 assume that $\mathfrak{A}\mathfrak{D} = \mathfrak{A}\mathfrak{D}$ and also that

$$(\delta, \mathfrak{D})(a^{t_1}b^{t_2}c^{t_3}d^{t_4}, \mathfrak{A}) = (na^{t_1}b^{t_2}c^{t_3}d^{t_4}, \mathfrak{A})(\delta, \mathfrak{D})$$

with $n \in N$.

8.2 The 3-dimensional, 2-step infra-nilmanifolds

As in the previous chapter we summarize our results in tables, but now the situation is much more delicate because of the presence of the parameters k_1, k_2, k_3, k_4. For a given family the k_2, k_3, k_4 are always fixed integers, but k_1 is a strict positive integer ($k_1 > 0$) and in most occasions it has to satisfy some extra conditions. In [12] k_1 is replaced by k and we use the same convention. It is clear that for each value of $k_1 > 0$ we have another almost-Bieberbach group and so also another infra-nilmanifold.

Therefore in each dimension, we present for each possible holonomy group F a table consisting of all families of almost-Bieberbach groups which have F as holonomy group. In the first two columns, we present the families and summarize the conditions on k (if there are any). In the following columns we summarize as before all possibilities concerning the Anosov relation for the corresponding manifolds. For the enumeration of the almost-Bieberbach groups and more details we refer to [12].

For the nilmanifolds in dimension 3 we have the following table.

1				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
1	/	1	0	0

This means that for each $k > 0$ we have a nilmanifold for which of course the Anosov theorem holds.
For the infra-nilmanifolds with $\mathbb{Z}_2$ as holonomy group we already have an interesting result.

$\mathbb{Z}_2$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
2	$k \equiv 0 \bmod 2$	0	1	0
4	$k \equiv 0 \bmod 2$	1	0	0

Indeed, this table indicates that for each $k > 0$ such that $k \equiv 0 \bmod 2$ we have one infra-nilmanifold for which the Anosov theorem holds (Family 4) and one infra-nilmanifold for which it does not hold (Family 2). Family 2 is another, much more simple, example that in general Proposition 5.12 does not hold for infra-nilmanifolds. To prove this, we use the techniques described in the previous section to find that α can be viewed as $(c^{\frac{1}{2}}, \mathfrak{A})$ such that $\mathfrak{A}_*$ is determined by the matrix

$$T_{\mathfrak{A}_*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now let f be the map on this infra-nilmanifold M induced by $(1, \mathfrak{D})$, where $\mathfrak{D}_*$ is given by

$$T_{\mathfrak{D}_*} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

It is easy to verify that $(1, \mathfrak{D})$ really induces a map on M, since, by using the matrices above, one computes that $\mathfrak{D}\mathfrak{A} = \mathfrak{A}\mathfrak{D}$ and hence

$$\begin{aligned} (1, \mathfrak{D})(c^{\frac{1}{2}}, \mathfrak{A}) &= (\mathfrak{D}(c^{\frac{1}{2}}), \mathfrak{D}\mathfrak{A}) \\ &= (c^{\frac{-1}{2}}, \mathfrak{D}\mathfrak{A}) \\ &= (c^{-1}, \mathcal{E})(c^{\frac{1}{2}}, \mathfrak{A}\mathfrak{D}) \\ &= (c^{-1}, \mathcal{E})(c^{\frac{1}{2}}, \mathfrak{A})(1, \mathfrak{D}). \end{aligned}$$

(Here $\mathcal{E}$ denotes the identity endomorphism of G). For this map the Anosov relation does not hold since $\det(I_3 - T_{\mathfrak{D}_*}) = -4$, while $(\det(I_3 - T_{\mathfrak{A}_*} T_{\mathfrak{D}_*}) = 4$.

Note that $\mathfrak{A}$ generates the holonomy group of M and therefore $T_{\mathfrak{A}_*}$ may be seen as the holonomy representation of $\mathfrak{A}$. In the sequel, we will slightly abuse notation by using the familiar $T_*(\mathfrak{A})$ instead of $T_{\mathfrak{A}_*}$ (without formally defining the holonomy representation) and $\mathfrak{D}_*$ instead of $T_{\mathfrak{D}_*}$.

To verify the statement for Family 4, we need a description of all suitable affine endomorphisms $(\delta,\mathfrak{D}) \in G \rtimes \mathrm{Endo}(G)$. For such a suitable affine endomorphism, we have for any $(t,\mathfrak{T})$ that

$$\theta(t,\mathfrak{T})(\delta,\mathfrak{D}) = (\delta,\mathfrak{D})(t,\mathfrak{T}).$$

Supposing that $\theta(t,\mathfrak{T}) = (t_1,\mathfrak{T}_1)$, this implies that

$$(t_1\mathfrak{T}_1(\delta),\mathfrak{T}_1\mathfrak{D}) = (\delta\mathfrak{D}(t),\mathfrak{D}\mathfrak{T})$$

and as in the flat case, we obtain the two types of restrictions: one on the linear parts and one on the translational parts. For family 4 there is only one generator with non-trivial linear part, namely $(a^{\frac{1}{2}},\mathfrak{A})$ with $T_*(\mathfrak{A}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Because of Proposition 2.6, we only have to consider the $(\delta,\mathfrak{D})$ for which $\mathfrak{D}_*$ commutes with $T_*(\mathfrak{A})$. This implies that

$$\mathfrak{D}_* = \begin{pmatrix} d_{22}d_{33} & 0 & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$$

where all the entries of $\mathfrak{D}_*$ have to be integers. Actually this last claim, follows from the fact that the corresponding homomorphism has to map the lattice N into itself. If this would not be the case, then again because of Proposition 2.6, the Anosov theorem holds and we can forget about this case.

The restriction on the translational part for this case becomes

$$na^{\frac{1}{2}}\mathfrak{A}(\delta) = \delta\mathfrak{D}(a^{\frac{1}{2}})$$

with $n \in N$ (and so $t_1 = na^{\frac{1}{2}}$). Equivalently, we have that

$$\delta\mathfrak{D}(a^{\frac{1}{2}})(a^{\frac{1}{2}}\mathfrak{A}(\delta))^{-1} = \delta\mathfrak{D}(a^{\frac{1}{2}})\mathfrak{A}(\delta^{-1})a^{\frac{-1}{2}} \in N.$$

The above statement can be verified by using the faithful matrix representation λ to rewrite the element. Indeed, the statement holds if and only if this matrix representation can be written as $\lambda(a^{z_1}b^{z_2}c^{z_3})$ with $z_1,z_2,z_3 \in \mathbb{Z}$. Taking $\delta = a^{d_1}b^{d_2}c^{d_3}$, we obtain

$$\begin{aligned}\lambda(\delta\mathfrak{D}(a^{\frac{1}{2}})\mathfrak{A}(\delta^{-1})a^{\frac{-1}{2}}) &= \begin{pmatrix} 1 & ld_2 & \frac{l}{4}(1-d_{22}) & -\frac{ld_2}{2}(1+4d_1)+2d_3 \\ 0 & 1 & 0 & \frac{1}{2}(-1+d_{22}) \\ 0 & 0 & 1 & 2d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \lambda(a^{\frac{-1+d_{22}}{2}})\lambda(b^{2d_2})\lambda(c^{\frac{ld_2}{2}(-2+d_{22}-4d_1)+2d_3}).\end{aligned}$$

So to satisfy the restrictions on the translational parts, one of the implications is that $d_{22} \in 1 + 2\mathbb{Z}$. There are also restrictions on δ, but these will have no influence on the validity of the Anosov theorem and so we disregard them.

If we now calculate the determinants occurring in Theorem 2.5, we obtain that $\det(I_3 - \mathfrak{D}_*) = (1 - d_{22}d_{33})(1 - d_{22})(1 - d_{33})$ and $\det(I_3 - T_*(\mathfrak{A})\mathfrak{D}_*) = (1 + d_{22}d_{33})(1 - d_{22})(1 + d_{33})$. To possibly obtain a counterexample, these determinants should have different signs, so we only have to focus on the first and the last factor. Since d_{22} can not be zero, we can easily verify that $(1 - d_{22}d_{33})(1 - d_{33})$ and $(1 + d_{22}d_{33})(1 + d_{33})$ have the same sign. Indeed, this certainly holds if $d_{33} = 0$ or ± 1 and can be checked if $|d_{33}| > 1$ since $d_{22} \neq 0$. So we may conclude that the Anosov theorem holds for all the manifolds belonging to Family 4.

In the following 3 tables we present the results for the other infra-nilmanifolds with cyclic holonomy group in dimension 3. All these results follow directly from Theorem 5.4.

$\mathbb{Z}_3$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
13	$k \equiv 0 \bmod 3$	2	0	0
	$k \not\equiv 0 \bmod 3$	1	0	0

$\mathbb{Z}_4$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
10	$k \equiv 0 \bmod 2$	1	0	0
	$k \equiv 0 \bmod 4$	1	0	0

$\mathbb{Z}_6$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
16	$k \equiv 0 \bmod 6$	2	0	0
	$k \equiv 2 \bmod 6$	1	0	0
	$k \equiv 4 \bmod 6$	1	0	0

The last family we need to examine, are infra-nilmanifolds with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as holonomy group. These manifolds are not covered by Theorem 6.10, which is only dealing with flat manifolds. Intriguingly we also obtain that the Anosov theorem holds.

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
8	$k \equiv 0 \bmod 4$	1	0	0

To prove this, we first have to describe the generators with non-trivial part of the almost-Bieberbach group of Family 8 as affine maps. These are $(c^{\frac{1}{2}}, \mathfrak{A}_1)$ and $(a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{k}{8}}, \mathfrak{A}_2)$ with

$$T_*(\mathfrak{A}_1) = \begin{pmatrix} 1 & \frac{k}{2} & \frac{-k}{2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad T_*(\mathfrak{A}_2) = \begin{pmatrix} -1 & \frac{-k}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

As argued in the previous chapter, to determine all suitable affine endomorphisms $(\delta, \mathfrak{D})$, we again have to consider 16 possibilities for $\mathfrak{D}$ or equivalently for $\mathfrak{D}_*$. These possibilities for $\mathfrak{D}_*$ are given in the table below.

$\mathfrak{D}_1 = \mathfrak{D}_1\mathfrak{A}_1$ $\mathfrak{D}_1 = \mathfrak{D}_1\mathfrak{A}_2$ $(\mathfrak{D}_1)_* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathfrak{D}_2 = \mathfrak{D}_2\mathfrak{A}_1$ $\mathfrak{A}_1\mathfrak{D}_2 = \mathfrak{D}_2\mathfrak{A}_2$ $(\mathfrak{D}_2)_* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\mathfrak{D}_3 = \mathfrak{D}_3\mathfrak{A}_1$ $\mathfrak{A}_2\mathfrak{D}_3 = \mathfrak{D}_3\mathfrak{A}_2$ $(\mathfrak{D}_3)_* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathfrak{D}_4 = \mathfrak{D}_4\mathfrak{A}_1$ $\mathfrak{A}_1\mathfrak{A}_2\mathfrak{D}_4 = \mathfrak{D}_4\mathfrak{A}_2$ $(\mathfrak{D}_4)_* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\mathfrak{A}_1\mathfrak{D}_5 = \mathfrak{D}_5\mathfrak{A}_1$ $\mathfrak{D}_5 = \mathfrak{D}_5\mathfrak{A}_2$ $(\mathfrak{D}_5)_* = \begin{pmatrix} 0 & \frac{k}{4}(d_{32} - d_{22}) & 0 \\ 0 & d_{22} & 0 \\ 0 & d_{32} & 0 \end{pmatrix}$	$\mathfrak{A}_1\mathfrak{D}_6 = \mathfrak{D}_6\mathfrak{A}_1$ $\mathfrak{A}_1\mathfrak{D}_6 = \mathfrak{D}_6\mathfrak{A}_2$ $(\mathfrak{D}_6)_* = \begin{pmatrix} 0 & 0 & \frac{k}{4}(d_{33} - d_{23}) \\ 0 & 0 & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}$

$\mathfrak{A}_1\mathfrak{D}_7 = \mathfrak{D}_7\mathfrak{A}_1$ $\mathfrak{A}_2\mathfrak{D}_7 = \mathfrak{D}_7\mathfrak{A}_2$ $(\mathfrak{D}_7)_* = \begin{pmatrix} d_{22}d_{33} & \frac{k}{4}d_{22}(d_{33}-1) & \frac{k}{4}d_{33}(d_{22}-1) \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$	
$\mathfrak{A}_1\mathfrak{D}_8 = \mathfrak{D}_8\mathfrak{A}_1$ $\mathfrak{A}_1\mathfrak{A}_2\mathfrak{D}_8 = \mathfrak{D}_8\mathfrak{A}_2$ $(\mathfrak{D}_8)_* = \begin{pmatrix} -d_{23}d_{32} & \frac{k}{4}d_{32}(1-d_{23}) & \frac{k}{4}d_{23}(d_{32}-1) \\ 0 & 0 & d_{23} \\ 0 & d_{32} & 0 \end{pmatrix}$	
$\mathfrak{A}_2\mathfrak{D}_9 = \mathfrak{D}_9\mathfrak{A}_1$ $\mathfrak{D}_9 = \mathfrak{D}_9\mathfrak{A}_2$ $(\mathfrak{D}_9)_* = \begin{pmatrix} 0 & d_{12} & 0 \\ 0 & 0 & 0 \\ 0 & d_{32} & 0 \end{pmatrix}$	$\mathfrak{A}_2\mathfrak{D}_{10} = \mathfrak{D}_{10}\mathfrak{A}_1$ $\mathfrak{A}_1\mathfrak{D}_{10} = \mathfrak{D}_{10}\mathfrak{A}_2$ $(\mathfrak{D}_{10})_* = \begin{pmatrix} 0 & d_{12} & \frac{k}{4}d_{33} \\ 0 & 0 & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$
$\mathfrak{A}_2\mathfrak{D}_{11} = \mathfrak{D}_{11}\mathfrak{A}_1$ $\mathfrak{A}_2\mathfrak{D}_{11} = \mathfrak{D}_{11}\mathfrak{A}_2$ $(\mathfrak{D}_{11})_* = \begin{pmatrix} 0 & 0 & d_{13} \\ 0 & 0 & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$	$\mathfrak{A}_2\mathfrak{D}_{12} = \mathfrak{D}_{12}\mathfrak{A}_1$ $\mathfrak{A}_1\mathfrak{A}_2\mathfrak{D}_{12} = \mathfrak{D}_{12}\mathfrak{A}_2$ $(\mathfrak{D}_{12})_* = \begin{pmatrix} 0 & \frac{k}{4}d_{32} & d_{13} \\ 0 & 0 & d_{23} \\ 0 & d_{32} & 0 \end{pmatrix}$
$\mathfrak{A}_1\mathfrak{A}_2\mathfrak{D}_{13} = \mathfrak{D}_{13}\mathfrak{A}_1$ $\mathfrak{D}_{13} = \mathfrak{D}_{13}\mathfrak{A}_2$ $(\mathfrak{D}_{13})_* = \begin{pmatrix} 0 & d_{12} & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathfrak{A}_1\mathfrak{A}_2\mathfrak{D}_{14} = \mathfrak{D}_{14}\mathfrak{A}_1$ $\mathfrak{A}_1\mathfrak{D}_{14} = \mathfrak{D}_{14}\mathfrak{A}_2$ $(\mathfrak{D}_{14})_* = \begin{pmatrix} 0 & d_{12} & \frac{-k}{4}d_{23} \\ 0 & 0 & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$

$\mathfrak{A}_1\mathfrak{A}_2\mathfrak{D}_{15} = \mathfrak{D}_{15}\mathfrak{A}_1$ $\mathfrak{A}_2\mathfrak{D}_{15} = \mathfrak{D}_{15}\mathfrak{A}_2$ $(\mathfrak{D}_{15})_* = \begin{pmatrix} 0 & 0 & d_{13} \\ 0 & d_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathfrak{A}_1\mathfrak{A}_2\mathfrak{D}_{16} = \mathfrak{D}_{16}\mathfrak{A}_1$ $\mathfrak{A}_1\mathfrak{A}_2\mathfrak{D}_{16} = \mathfrak{D}_{16}\mathfrak{A}_2$ $(\mathfrak{D}_{16})_* = \begin{pmatrix} 0 & 0 & d_{13} \\ 0 & 0 & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$

Before we analyze the restrictions on the translational parts, we can already drop the cases $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4, \mathfrak{D}_9, \mathfrak{D}_{14}, \mathfrak{D}_{16}$ since for these cases the determinants of Theorem 2.5 are all equal to 1. Secondly, because of Proposition 2.6 we know that the Anosov relation holds for the case of $\mathfrak{D}_{12}$. This reduces our investigation to 8 possibilities. Thirdly, in the case $\mathfrak{D}_8$, the determinants of Theorem 2.5 are all equal to $(1-d_{23}d_{32})^2 + (1+d_{23}d_{32})^2$. So again the Anosov theorem holds since all these determinants are strictly positive. So, only 7 cases remain.

From these 7 cases we can eliminate 6 more since they are not compatible with the respective restrictions on the translational parts. For example for $\mathfrak{D}_9$, we have that $\delta\mathfrak{D}(c^{\frac{1}{2}})\mathfrak{A}_2(\delta^{-1})(a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{k}{8}})^{-1}$ is not an element of N since we can rewrite this element as

$$a^{\frac{-1}{2}}b^{\frac{-1}{2}+2d_2}c^{\frac{k}{8}(1+4d_1-16d_1d_2-8d_2+16d_3)}.$$

Note that again we have taken $\delta = (d_1, d_2, d_3)^t$. Similar arguments hold for $\mathfrak{D}_5, \mathfrak{D}_6, \mathfrak{D}_{10}, \mathfrak{D}_{11}, \mathfrak{D}_{13}$ and $\mathfrak{D}_{15}$.

So to check the validity of the Anosov theorem, we only have to focus on the case $\mathfrak{D}_7$. From analyzing the translational restrictions in the way explained before, we obtain that d_{22} and d_{33} must be odd integers. This is very useful since the 4 determinants we have to consider are

$$\begin{aligned}
\det(I_3 - (\mathfrak{D}_7)_*) &= (1-d_{22}d_{33})(1-d_{22})(1-d_{33}),\\
\det(I_3 - T_*(\mathfrak{A}_1)(\mathfrak{D}_7)_*) &= (1-d_{22}d_{33})(-1-d_{22})(-1-d_{33}),\\
\det(I_3 - T_*(\mathfrak{A}_2)(\mathfrak{D}_7)_*) &= (-1-d_{22}d_{33})(1-d_{22})(-1-d_{33}),\\
\det(I_3 - T_*(\mathfrak{A}_1\mathfrak{A}_2)(\mathfrak{D}_7)_*) &= (-1-d_{22}d_{33})(-1-d_{22})(1-d_{33}).
\end{aligned}$$

Now, Lemma 6.9 implies that the Anosov theorem holds for this case, which proves our statement concerning Family 8.

To summarize, we may conclude that the Anosov theorem holds for all, except one, families of infra-nilmanifolds in dimension 3. This result depends as before on the respective holonomy groups, but also on the structure of the universal covering Lie group which results in extra restrictions on the suitable linear parts.

The influence of the structure of the universal covering Lie group G is maximal if G is a filiform Lie group. An n-dimensional Lie group is called filiform if it is $(n-2)$-step nilpotent. Well for such a G we have a lot of structure on the affine endomorphisms and so also on the matrices we derive from them. Indeed these are $(n \times n)$-matrices where the first $n-1$ columns are completely determined by the last column and the structure of G. Obviously, one may expect that this also influences the signs of the determinants of Theorem 2.5. This leads to the following questions.

Question 8.2. *What can we say about the validity of the Anosov theorem for (orientable) infra-nilmanifolds with filiform universal covering Lie group? Or more generally: can we capture the relationship between the validity of the Anosov theorem and the structure of the universal covering Lie group?*

8.3 The 4-dimensional, 2-step infra-nilmanifolds

From dimension 4 onwards the classification is slightly more complex, since there are families which are subdivided into a finite number of subfamilies depending on the action of the generators with non-trivial linear part on the generator d (i.e. on the commutator subgroup) of the lattice N.

In order to be able to present our results, we need to agree on how we refer to a specific subfamily of a given family, say F (here F denotes a natural number as in the 3-dimensional case). An almost-Bieberbach group of Family F will be generated by a, b, c, d together with at most 3 generators with non-trivial linear part denoted by α, β and γ. If a family F is subdivided into several subfamilies, then these subfamilies are determined by the action of α, β and γ on d. Since such a generator acts or trivially on d or by inversion on d, we will denote a subfamily by

$F\ (\pm 1, \pm 1, \ldots)$ (where there are as many ± 1's as there are generators with non-trivial linear part), where the first ± 1 stands for the action of α on d (where of course 1 is used to denote the trivial action and -1 to indicate the non-trivial action). The second ± 1, if needed, is used to denote the action of β and analogously, if necessary the third one is used for γ. If the action on d is trivial for all generators $\alpha, \ldots$ we will omit this notation. Again we refer to [12] for the necessary details.

Just like we treated the flat manifolds in Chapter 7, we will also divide the infra-nilmanifolds into two groups: the ones with an abelian holonomy group and the ones with a non-abelian holonomy group. Of course, similar arguments as the ones used in Chapter 7 can be used to investigate both groups. Moreover, we can already mention that the investigation of the second group is very interesting since we will obtain that the Anosov theorem holds for all orientable infra-nilmanifolds of this group.

8.3.1 Abelian holonomy group

The majority of the infra-nilmanifolds in this section are covered by Theorem 5.4 (the infra-nilmanifolds with holonomy group $\mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_6$) and by Proposition 3.7 (the non-orientable infra-nilmanifolds). The only two classes, which need to be examined are the infra-nilmanifolds with holonomy group $\mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The calculations that need to be done here, are completely similar to the ones we did in the 3-dimensional case and for brevity we omit them. The table for the nilmanifolds is completely similar as the one in dimension 3.

1				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
1	/	1	0	0

For the infra-nilmanifolds with $\mathbb{Z}_2$ as holonomy group we obtain the following table.

$\mathbb{Z}_2$					
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
2	$k \equiv 0 \bmod 2$	0	0	1
3	$k \equiv 0 \bmod 2$	0	1	0
4	/	0	1	0
	$k \equiv 0 \bmod 2$	0	1	0
4 (-1)	/	0	0	1
5	/	0	1	0
6	/	0	0	1
7	/	0	0	2
7 (-1)	/	2	1	0
	$k \equiv 0 \bmod 2$	0	1	0
8	/	0	0	1
9	/	0	0	1
9 (-1)	/	1	1	0

To be precise for the Families $7(-1)$ and $9(-1)$, we were able to construct a counter example for the Bieberbach groups labelled $(k, 0, 0, 0)$.

We want to focus on two types of infra-nilmanifolds of Family $7(-1)$: the ones belonging to the almost-Bieberbach group labelled $(k, 0, 0, 0)$ and $(0, k, 0, 0)$. For both of these almost-Bieberbach groups the generator with non-trivial linear part is given by $(c^{\frac{1}{2}}, \mathfrak{A})$ with $T_*(\mathfrak{A}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. This implies that at first sight they admit the same suitable affine endomorphisms $(\delta, \mathfrak{D})$ with

$$\mathfrak{D}_* = \begin{pmatrix} d_{11} & 0 & d_{13} & 0 \\ 0 & d_{22} & 0 & d_{24} \\ 0 & 0 & d_{33} & 0 \\ 0 & d_{42} & 0 & d_{44} \end{pmatrix}$$

such that all entries are integers; $d_{24} \in 2\mathbb{Z}$ and $d_{44} \in 1 + 2\mathbb{Z}$.(As before we do not need to specify δ and we may assume that $\mathfrak{A}\mathfrak{D} = \mathfrak{D}\mathfrak{A}$). But the extra restrictions (coming from the nilpotent covering group) on the first column of the linear part $\mathfrak{D}_*$ differ. Since for the group labelled $(k, 0, 0, 0)$ we have that $[b, a] = d^k, [c, a] = 1, [c, b] = 1$ and so

$d_{11} = d_{22}d_{33}$. If we then take $\delta = 1$ and $\mathfrak{D}_* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&3&0 \\ 0&0&0&3 \end{pmatrix}$, we have that the Anosov relation does not hold for the map induced by this affine endomorphism $(\delta, \mathfrak{D})$ since $\det(I_4 - \mathfrak{D}_*) = 4$ and $\det(I_4 - T_*(\mathfrak{A})\mathfrak{D}_*) = -8$.

On the other hand for the group labelled $(0, k, 0, 0)$, we have that $[b, a] = 1, [c, a] = 1, [c, b] = d^{2k}$ and so $d_{11} = d_{33}d_{44}$. If we then again calculate the determinants of Theorem 2.5, we obtain that $\det(I_4 - \mathfrak{D}_*) = (1 - d_{33}d_{44})(1 - d_{33})(1 - d_{22})(1 - d_{44})$ and $\det(I_4 - T_*(\mathfrak{A})\mathfrak{D}_*) = (1 + d_{33}d_{44})(1 + d_{33})(1 - d_{22})(1 - d_{44})$. Since d_{44} can not be zero and d_{33} is an integer, one can as before verify that these determinants have the same sign. So Theorem 2.5 implies that the Anosov theorem holds for the corresponding infra-nilmanifold.

Family $7(-1)$ clearly demonstrates that the answer to Question 8.2 will in general be far from trivial since the relationship can be very delicate.

Finally, we also want to briefly present the other counter examples by giving for each family the translational part δ and the $\mathfrak{D}_*$ of an affine endomorphism $(\delta, \mathfrak{D})$ which induces a map on the respective infra-nilmanifold that not satisfies the Anosov relation. We only focus on the orientable infra-nilmanifolds and for both cases of the Families 4 and 7 we can respectively use the same counter example.

Family	δ	$\mathfrak{D}_*$	Family	δ	$\mathfrak{D}_*$
3	1	$\begin{pmatrix} -1&0&0&0 \\ 0&1&0&1 \\ 0&0&0&0 \\ 0&2&0&1 \end{pmatrix}$	4	1	$\begin{pmatrix} -1&0&0&0 \\ 0&1&0&1 \\ 0&0&3&0 \\ 0&2&0&1 \end{pmatrix}$
5	1	$\begin{pmatrix} -1&0&0&0 \\ 0&0&1&2 \\ 0&1&0&2 \\ 0&1&1&3 \end{pmatrix}$	7 (-1)	1	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&3&0 \\ 0&0&0&3 \end{pmatrix}$
9 (-1)	1	$\begin{pmatrix} 0&0&0&0 \\ 0&1&-1&0 \\ 0&-1&1&0 \\ 0&0&0&3 \end{pmatrix}$			

For the infra-nilmanifolds with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as holonomy group we obtain the following table.

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
11	$k \equiv 0 \bmod 2$	0	0	1
13	$k \equiv 0 \bmod 2$	0	0	1
14	$k \equiv 0 \bmod 2$	0	0	2
	$k \not\equiv 0 \bmod 2$	0	0	1
14 (-1,1)	/	0	0	1
	$k \equiv 0 \bmod 2$	0	0	1
15	$k \equiv 0 \bmod 2$	0	0	1
18 (1,-1)	$k \equiv 0 \bmod 2$	0	0	1
19 (-1,1)	/	0	0	1
26 (-1,1)	/	0	0	1
27 (1,-1)	$k \equiv 0 \bmod 2$	1	0	0
29 (-1,1)	/	0	0	1
	$k \equiv 0 \bmod 2$	0	0	1
29 (1,-1)	/	0	3	0
29 (-1,-1)	/	0	0	2
30 (1,-1)	/	0	1	0
31 (-1,1)	/	0	0	1
32 (1,-1)	$k \equiv 0 \bmod 2$	2	0	0
33 (-1,1)	/	0	0	1
33 (1,-1)	/	0	2	0
33 (1,-1)	$k \equiv 0 \bmod 2$	0	1	0
33 (-1,-1)	/	0	0	1
34 (1,-1)	/	1	0	0
36 (-1,1)	/	0	0	1
37 (1,-1)	$k \equiv 0 \bmod 2$	1	0	0
41 (1,-1)	/	2	0	0
43 (1,-1)	$k \equiv 0 \bmod 2$	1	0	0
	$k \not\equiv 0 \bmod 2$	0	1	0
45 (1,-1)	$k \equiv 0 \bmod 2$	2	0	0

For these infra-nilmanifolds we want to focus to Family $43(1,-1)$ since this is the first case where the validity of the Anosov theorem depends explicitly on the value of k. The almost Bieberbach groups in this case, are generated by $a, b, c, d, \alpha, \beta$. The α, β are respectively equivalent to $(c^{\frac{1}{2}}, \mathfrak{A}_1), (b^{\frac{1}{2}}, \mathfrak{A}_2)$ with

$$T_*(\mathfrak{A}_1) = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad T_*(\mathfrak{A}_2) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

The role of k is determined by $[b, a] = d^k, [c, a] = d^k, [c, b] = d^{-k}$.

Similar calculations as in Section 8.2 lead to the fact that there are only two types of affine endomorphisms which can be used to construct a counter example. The other types do not exist, because of the restrictions on the translational parts, or lead to determinants which all have the same sign. The two types are $(\delta_5, \mathfrak{D}_5)$ with

$$(\mathfrak{D}_5)_* = \begin{pmatrix} 0 & d_{12} & d_{13} & -d_{12} + d_{13} \\ 0 & d_{22} & d_{23} & -d_{22} + d_{23} \\ 0 & d_{22} & 2d_{22} - d_{23} & d_{22} - d_{23} \\ 0 & -d_{22} & d_{12} - d_{13} - d_{23} & d_{12} - d_{13} + d_{22} - d_{23} \end{pmatrix}$$

(which results from $\mathfrak{A}_1\mathfrak{D}_5 = \mathfrak{D}_5\mathfrak{A}_1$; $\mathfrak{D}_5 = \mathfrak{D}_5\mathfrak{A}_2$) and $(\delta_6, \mathfrak{D}_6)$ with

$$(\mathfrak{D}_6)_* = \begin{pmatrix} 0 & d_{12} & d_{13} & d_{12} - d_{13} \\ 0 & d_{22} & d_{23} & d_{22} - d_{23} \\ 0 & -d_{22} + 2d_{23} & d_{23} & -d_{22} + d_{23} \\ 0 & -d_{12} + d_{13} - d_{22} & -d_{23} & -d_{12} + d_{13} - d_{22} + d_{23} \end{pmatrix}$$

(which results from $\mathfrak{A}_1\mathfrak{D}_6 = \mathfrak{D}_6\mathfrak{A}_1$; $\mathfrak{A}_1\mathfrak{D}_6 = \mathfrak{D}_6\mathfrak{A}_2$). The calculations and conclusions are the same for both cases, so we proceed with the first one. Rewriting the restrictions on the translational parts and using $\delta = (d_1, d_2, d_3, d_4)^t, x = \frac{-d_{22}+d_{23}}{2} + d_1 - d_2, y = \frac{d12-d13}{2} + 2d_1 + 2d_3$, we obtain

$$\delta\mathfrak{D}(c^{\frac{1}{2}})\mathfrak{A}(\delta^{-1})c^{\frac{-1}{2}} = a^x b^{-x} c^{\frac{-1}{2}+x+y} d^{-y-\frac{k}{2}x(3x-2y)}.$$

This can only be an element of N if all the exponents are integers. So x and $\frac{-1}{2} + y$ must be integers and therefore we have that $\frac{k}{2}x(3x-2y)$ must be an element of $\frac{1}{2} + \mathbb{Z}$. Now $x(3x-2y)$ is also an integer because of the restrictions on x, y. Therefore if k is even, the exponent of d can never be an integer and therefore we can not construct a counter example. On the other hand, if k is odd and we carefully choose x

and y, then we can construct a counter example. For instance the map induced by $(\delta, \mathfrak{D})$ with

$$\delta = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{D}_* = \begin{pmatrix} 0 & 9 & 1 & -8 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 6 & 6 \end{pmatrix}$$

does not satisfy the Anosov relation since $\det(I_4 - \mathfrak{D}_*) = -3$ and $\det(I_4 - \mathfrak{A}_*\mathfrak{D}_*) = 5$.

Again we end with briefly describing all counter examples. Note that for the three cases of Family $29(1,-1)$ we can use the same counter example. The first entry of the second line corresponds with the almost-Bieberbach group labelled $(k, 0, 0, 0, 0)$ and the second one with the one labelled $(k, 0, 1, 0, 0)$. In the last line we consider the case $k \equiv 0 \bmod 2$.

Family	δ	$\mathfrak{D}_*$	Family	δ	$\mathfrak{D}_*$
29 (1,-1)	1	$\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&3&0 \\ 0&0&0&3 \end{pmatrix}$	30 (1,-1)	1	$\begin{pmatrix} 0&0&-3&0 \\ 0&0&0&0 \\ 0&0&3&0 \\ 0&0&0&3 \end{pmatrix}$
33(1,-1)	$a^{\frac{1}{4}}$	$\begin{pmatrix} 0&3k&0&2 \\ 0&2&0&0 \\ 0&2&0&0 \\ 0&0&0&3 \end{pmatrix}$	33(1,-1)	$a^{\frac{1}{4}}$	$\begin{pmatrix} 0&-2+3k&0&2 \\ 0&2&0&0 \\ 0&2&0&0 \\ 0&0&0&3 \end{pmatrix}$
33(1,-1)	$a^{\frac{1}{4}}$	$\begin{pmatrix} 0&-2+6k&0&2 \\ 0&4&0&0 \\ 0&4&0&0 \\ 0&0&0&3 \end{pmatrix}$			

For the remaining tables, we did not have to make explicit calculations, so we just present the results.

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
56 (1,-1,1)	$k \equiv 0 \bmod 2$	0	0	1
60 (1,-1,1)	/	0	0	1
61 (-1,1,1)	$k \equiv 0 \bmod 2$	0	0	2
	$k \not\equiv 0 \bmod 2$	0	0	2
62 (-1,1,1)	$k \equiv 0 \bmod 2$	0	0	1

$\mathbb{Z}_4$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
75	$k \equiv 0 \bmod 2$	1	0	0
	$k \equiv 0 \bmod 4$	1	0	0
76	/	1	0	0
76	$k \equiv 0 \bmod 2$	1	0	0
77	$k \equiv 0 \bmod 2$	1	0	0
79	$k \equiv 0 \bmod 2$	2	0	0
80	/	1	0	0
81	$k \equiv 0 \bmod 2$	0	0	2
	$k \equiv 0 \bmod 4$	0	0	1
82	$k \equiv 0 \bmod 2$	0	0	2
	$k \not\equiv 0 \bmod 2$	0	0	1
	$k \equiv 0 \bmod 4$	0	0	1
	$k \equiv 2 \bmod 4$	0	0	1

$\mathbb{Z}_2 \oplus \mathbb{Z}_4$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
85	$k \equiv 0 \bmod 2$	0	0	2
86	$k \equiv 0 \bmod 2$	0	0	2
88	$k \equiv 0 \bmod 2$	0	0	2

$\mathbb{Z}_3$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
143	$k \equiv 0 \bmod 3$	2	0	0
	$k \not\equiv 0 \bmod 3$	1	0	0
144	/	1	0	0
	$k \equiv 0 \bmod 3$	1	0	0
146	/	2	0	0

Note that we added Family 143 of [18] which is missing in [12].

$\mathbb{Z}_6$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
147	$k \equiv 0 \bmod 6$	0	0	2
	$k \equiv 2 \bmod 6$	0	0	1
	$k \equiv 4 \bmod 6$	0	0	1
148	$k \equiv 0 \bmod 2$	0	0	2
168	$k \equiv 0 \bmod 6$	2	0	0
	$k \equiv 2 \bmod 6$	1	0	0
	$k \equiv 4 \bmod 6$	1	0	0
169	/	1	0	0
172	$k \equiv 0 \bmod 2$	1	0	0
173	$k \equiv 0 \bmod 3$	2	0	0
	$k \equiv 1 \bmod 3$	1	0	0
	$k \equiv 2 \bmod 3$	1	0	0
174	$k \equiv 0 \bmod 3$	0	0	2
	$k \not\equiv 0 \bmod 3$	0	0	1

$\mathbb{Z}_2 \oplus \mathbb{Z}_6$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
176	$k \equiv 0 \bmod 6$	0	0	2
	$k \equiv 2 \bmod 6$	0	0	1
	$k \equiv 4 \bmod 6$	0	0	1

To summarize this section, we may say that these tables show that already many infra-nilmanifolds are covered by our results of part 2. Secondly, the relationship between the structure of the universal covering Lie group of a given infra-nilmanifold and the validity of the Anosov theorem for this manifold is very delicate. So answering Question 8.2 will not be straightforward. At least in general, but perhaps in restricted cases, such as the filiform Lie groups, the situation might become easier.

8.3.2 Non-abelian holonomy group

Part of the results in this section are again obtained by Proposition 3.7 concerning 2-step non-orientable infra-nilmanifolds. For the remaining orientable infra-nilmanifolds we find that the Anosov theorem always holds, which is rather surprising when we compare this with the analogous results in Chapter 7. The results for the infra-nilmanifolds with D_8 as holonomy group are summarized in the table below.

D_8				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
103 (1,-1)	$k \equiv 0 \bmod 2$	1	0	0
	$k \equiv 0 \bmod 4$	1	0	0
104 (1,-1)	$k \equiv 0 \bmod 2$	2	0	0
106 (1,-1)	$k \equiv 0 \bmod 2$	2	0	0
110 (1,-1)	/	2	0	0
114 (1,-1)	$k \equiv 0 \bmod 2$	0	0	2

To obtain these results we start by showing that we only have to focus on the first case of Family $103(1,-1)$, since for the other cases the suitable affine endomorphisms are very similar. This similarity leads to the fact that the determinants of Theorem 2.5 are exactly the same for all cases.

Let us first fix some notations: the generators with non-trivial linear part for the first case of Family $103(1,-1)$ are $(d^{\frac{1}{4}}, \mathfrak{A}_1)$ and $(c^{\frac{1}{2}}, \mathfrak{A}_2)$ with

$$T_*(\mathfrak{A}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_*(\mathfrak{A}_2) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we derive the matrices from the holonomy representation for the holonomy groups of the infra-nilmanifolds of Family $103(1,-1)$, Family $104(1,-1)$ and Family $106(1,-1)$, then we find that these matrices only differ in the last three elements of the first row. Of course we only compare elements which are generated in the same way in the respective holonomy groups. For instance for Family $104(1,-1)$, we have

$$T_*(\mathfrak{A}'_1) = \begin{pmatrix} 1 & k & 2-k & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_*(\mathfrak{A}'_2) = \begin{pmatrix} -1 & -2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which clearly implies that indeed the matrices can only differ in the first row for both these cases.

As demonstrated many times before, to calculate all possibilities for the matrices $\mathfrak{D}_*$ we need these matrices of the holonomy representation. Since the respective holonomy representations are very 'similar' for each of the respective infra-nilmanifolds, the same holds for the possibilities for the $\mathfrak{D}_*$ in each case. Namely, the 'corresponding' $\mathfrak{D}_*$ also only differ in the last three elements of the first row. By corresponding we mean here, belonging to the analogous homomorphism on the respective fundamental groups. This relies completely on the fact that the last three elements of the first column of the matrices $\mathfrak{D}_*$ are all zero.

Now, if we want to verify the validity of the Anosov theorem for one of the infra-nilmanifolds of the respective families, we have to calculate the determinants of Theorem 2.5 for all suitable affine endomorphisms $(\delta, \mathfrak{D})$. Let us fix an element x of D_8, then the matrices $T_*(x)$ from the respective holonomy representations only differ in the first row and the last three elements of the first column are always zero. If we then consider for each infra-nilmanifold the corresponding $(\delta, \mathfrak{D})$, the above implies that $\det(I_n - T_*(x)\mathfrak{D}_*)$ is always the same.

At first sight, the same does not hold for Family $110(1,-1)$ but we come back to this later on.

Now, let us proceed with Family $103(1,-1)$. To calculate all suitable affine endomorphisms, we have to consider 64 possibilities. Indeed, as explained before, to construct all possible homomorphisms, we have for each of the two generators of the holonomy group 8 possibilities since D_8 has 8 elements. From the 64 possibilities, there are 56 possibilities which lead to a matrix $\mathfrak{D}_*$ of the form

$$\begin{pmatrix} 0 & 0 & 0 & d_{14} \\ 0 & 0 & 0 & d_{24} \\ 0 & 0 & 0 & d_{34} \\ 0 & 0 & 0 & d_{44} \end{pmatrix}.$$

For these 56 possibilities, the determinants in Theorem 2.5 are all equal to $(1 - d_{44})$ and so the Anosov theorem holds. The remaining 8 possibilities result in the following $\mathfrak{D}_*$

$(\mathfrak{D}_1)_* = \begin{pmatrix} (d_{22})^2 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & d_{22} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	$(\mathfrak{D}_2)_* = \begin{pmatrix} -(d_{22})^2 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & -d_{22} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$
$(\mathfrak{D}_3)_* = \begin{pmatrix} (d_{22})^2 & 0 & 0 & 0 \\ 0 & 0 & -d_{22} & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	$(\mathfrak{D}_4)_* = \begin{pmatrix} -(d_{22})^2 & 0 & 0 & 0 \\ 0 & 0 & -d_{22} & 0 \\ 0 & -d_{22} & 0 & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$
$(\mathfrak{D}_5)_* = \begin{pmatrix} -2(d_{22})^2 & 0 & 0 & 0 \\ 0 & d_{22} & d_{22} & 0 \\ 0 & d_{22} & -d_{22} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	$(\mathfrak{D}_6)_* = \begin{pmatrix} 2(d_{22})^2 & 0 & 0 & 0 \\ 0 & d_{22} & -d_{22} & 0 \\ 0 & d_{22} & d_{22} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$
$(\mathfrak{D}_7)_* = \begin{pmatrix} 2(d_{22})^2 & 0 & 0 & 0 \\ 0 & d_{22} & d_{22} & 0 \\ 0 & -d_{22} & d_{22} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$	$(\mathfrak{D}_8)_* = \begin{pmatrix} -2(d_{22})^2 & 0 & 0 & 0 \\ 0 & d_{22} & -d_{22} & 0 \\ 0 & -d_{22} & -d_{22} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix}$

Note that the d_{ij} can be more specified by examining the restrictions on the translational parts and by considering the possibilities for the image of the lattice (under the homomorphism). We omit this since we do not need it to show the validity of the Anosov theorem.

If we calculate the determinants of Theorem 2.5 for each of these possibilities, we obtain that all determinants have the same form. Namely, if the element in the upper corner of $\mathfrak{D}_*$ is $\pm(d_{22})^2$, we have

$$(-1 + (d_{22})^2)(-1 + d_{44}) \cdot X$$

and in the other case we have

$$(-1 + 2(d_{22})^2)(-1 + d_{44}) \cdot X$$

with X always strict positive and possibly different for the different determinants. This clearly implies that for each of the respective cases that the determinants have the same sign, so we may conclude that the Anosov theorem indeed always holds.

The same conclusion holds for Family $110(1,-1)$ and this is obtained in the same way, but first we have to transform the matrices $T_*(\mathfrak{A}''_1), T_*(\mathfrak{A}''_2)$ to $PT_*(\mathfrak{A}''_1)P^{-1}, PT_*(\mathfrak{A}''_2)P^{-1}$ with

$$P = \begin{pmatrix} 2 & \frac{4+9k}{2} & \frac{-4-13k}{2} & \frac{4+9k}{2} \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

After that, we can apply exactly the same reasoning as before but we have to slightly adapt the 8 matrices $(\mathfrak{D}_i)_*$. Namely, since the structure of the lattice is different, the upper element of the $(\mathfrak{D}_i)_*$ is also different. However an analogue conclusion about the determinants holds.

It is important to note that the validity of the above results again heavily depend on the extra restrictions on the linear parts of the affine endomorphisms coming from the structure of the universal covering Lie group. Secondly, in the proof of our claims we did not use the restrictions on the translational parts.

Exactly the same arguments hold for the infra-nilmanifolds with S_3 and $\mathbb{Z}_2 \oplus S_3$ as holonomy group. The results are summarized in the tables below.

S$_3$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
158 (1,-1)	$k \equiv 0 \mod 3$	2	0	0
	$k \not\equiv 0 \mod 3$	1	0	0
159 (1,-1)	$k \equiv 0 \mod 3$	2	0	0
	$k \equiv 1 \mod 3$	1	0	0
	$k \equiv 2 \mod 3$	1	0	0
161 (1,-1)	$k \equiv 0 \mod 3$	2	0	0
	$k \equiv 1 \mod 3$	2	0	0
	$k \equiv 2 \mod 3$	2	0	0

$\mathbb{Z}_2 \oplus$ S$_3$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
184 (1,-1)	$k \equiv 0 \mod 6$	2	0	0
	$k \equiv 2$ mod	1	0	0
	$k \equiv 4$ mod	1	0	0

To conclude, the results obtained in this section are very intriguing since for the orientable infra-nilmanifolds in this section the Anosov theorem always holds. Since flat manifolds are a special type of infra-nilmanifolds, we know from Chapter 7 that this certainly does not hold for any infra-nilmanifold. So, in trying to generalize this, we have to focus on non-abelian Lie groups and we can reformulate Question 8.2. This restriction can be very useful since we already demonstrated that the relationship is very delicate.

Question 8.3. *Can we capture the relationship between the structure of the non-abelian universal covering Lie group and the validity of the Anosov theorem if we restrict ourselves to infra-nilmanifolds with non-abelian holonomy group?*

8.4 The 4-dimensional, 3-step infra-nilmanifolds

As in the previous section, a given family can be subdivided into subfamilies according to the action on the lattice generator d, therefore, we will use an analogous notation as before. Secondly some families are also subdivided into separate classes depending on some underlying almost-crystallographic group Q which is used. When this is needed for a specific family F, we will, without any further explanation, refer to this Q by means of the labels used in [12] . For instance $F[Q=(2\ell+1,1)](-1)$ means that in family F, there is a class of almost-Bieberbach groups labelled with $Q=(2\ell+1,1)$ for which the action of the generator with non-trivial linear part on d is equal to d^{-1}. For more details we refer to [12].[1]

Thirdly, we can no longer apply Proposition 3.7 on the non-orientable infra-nilmanifolds. However the main result of [19] implies that the infra-nilmanifolds that we consider in this section all admit an expanding map. So Theorem 3.6 still implies that the Anosov theorem does not hold for the non-orientable manifolds. For more details we refer to [19].

For the nilmanifolds we again we have a similar table as before.

1				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
1	/	1	0	0

[1] To be consistent with the previous tables,we slightly changed the notation used in [12]. To be precise, we replaced m by k and for the fourth family we replaced k by ℓ.

For the infra-nilmanifolds with $\mathbb{Z}_2$ as holonomy group we obtained the following results.

$\mathbb{Z}_2$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
2 (-1)	/	0	0	1
3[Q=(2ℓ+1,1)]	$k \equiv 0$ mod 2	1	0	0
3[Q=(2ℓ,1)]	/	1	0	0
3[Q=(2ℓ,0)]	$k \equiv 0$ mod 2	1	0	0
4[Q=(2ℓ)]	/	1	0	0
4[Q=(2ℓ)]	$k \equiv 0$ mod 2	2	0	0
4[Q=(2ℓ+1)]	/	1	0	0
4[Q=(2ℓ+1)]	$k \equiv 0$ mod 2	1	0	0
4[Q=(2ℓ+1)]	$k \equiv 0$ mod 4	1	0	0
4[Q=(ℓ)] (-1)	/	0	0	ℓ
5[Q=(2ℓ,0)]	/	1	0	0
5[Q=(2ℓ+1,0)]	/	1	0	0

The prove the results for the orientable manifolds we focus on the first case of Family 3. The calculations for the other cases are analogous. Applying the techniques explained in Section 8.1 we deduce from [12] that the generator with non-trivial linear part of the infra-nilmanifolds of the case under study is equivalent with $(d^{\frac{-1}{2}}, \mathfrak{A})$ such that

$$\mathfrak{A}_* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since the holonomy group is $\mathbb{Z}_2$, Proposition 2.6 implies that we only have to consider affine endomorphisms $(\delta, \mathfrak{D})$ such that $\mathfrak{D}\mathfrak{A} = \mathfrak{A}\mathfrak{D}$. This means that we may assume that for the suitable affine endomorphisms $(\delta, \mathfrak{D})$ we have that

$$\mathfrak{D}_* = \begin{pmatrix} d_{11} & 0 & d_{23} & 0 \\ d_{21} & d_{22} & d_{23} & d_{24} \\ -d_{21} & 0 & d_{22} - 2d_{23} & 0 \\ 0 & 2d_{43} & d_{43} & d_{44} \end{pmatrix}$$

with all entries integers. This $\mathfrak{D}_*$ does not take into account the restrictions coming from the structure of the universal covering group. Since the first column concerns the image of D and the second the image of C, we have that $d_{21} = d_{43} = 0$. Moreover d_{11}, d_{22} are completely determined by the fact that $[c,b] = d^k$ and $[b,a] = c^{2\ell+1}d^{k\ell}$ which are part of the relations we find in [12]. Indeed these relations implies that

$$
\begin{aligned}
& k\mathfrak{D}_*(D) = [\mathfrak{D}_*(C), \mathfrak{D}_*(B)] \\
\Leftrightarrow\ & kd_{11}D = [d_{22}C, d_{23}D + d_{23}C + (d_{22} - 2d_{23})A + d_{43}B] \\
\Leftrightarrow\ & kd_{11}D = d_{22}d_{43}[C,B] \\
\Leftrightarrow\ & kd_{11}D = kd_{22}d_{43}kD
\end{aligned}
$$

and so $d_{11} = d_{22}d_{43}$ since $k > 0$. Analogously we find that

$$
\begin{aligned}
(2\ell+1)\mathfrak{D}_*(C) &= d_{22}C \\
&= kd_{23}d_{44}D + (2\ell+1)(d_{22} - 2d_{23})d_{44}C.
\end{aligned}
$$

which implies that $d_{23} = 0$ or $d_{44} = 0$ and $d_{22} = (d_{22} - 2d_{23})d_{44}$.

The above results for the suitable affine endomorphisms are obtained by the restrictions on the linear parts and by the restrictions coming from the structure of the universal covering group. So there is still one type of restrictions left, namely these on the translational parts. For the first case of Family 3, we have $(\delta, \mathfrak{D})(d^{\frac{-1}{2}}, \mathfrak{A}) = (nd^{\frac{-1}{2}}, \mathfrak{A})(\delta, \mathfrak{D})$ with $n \in N$, so the restrictions on the translational parts are

$$\delta\mathfrak{D}(d^{\frac{-1}{2}})\mathfrak{A}(\delta^{-1})d^{\frac{1}{2}} \in N$$

By similar calculations as before this leads to the fact that d_{11} must be an odd integer and so d_{22} and d_{44} are non-zero integers.

To check the validity of the Anosov theorem, we use once more Theorem 2.5 and we have to determine the signs of the following two determinants.

$$
\begin{aligned}
\det(I_4 - \mathfrak{D}_*) &= (1 - d_{22}d_{44})(1 - d_{22})(1 - d_{22} + 2d_{23})(1 - d_{44}) \\
\det(I_4 - \mathfrak{A}_*\mathfrak{D}_*) &= (1 - d_{22}d_{44})(1 + d_{22})(1 - d_{22} + 2d_{23})(1 + d_{44})
\end{aligned}
$$

Because of the restrictions on the d_{22} and d_{44}, we easily obtain that the determinants have the same sign which implies that the Anosov theorem holds.

The infra-nilmanifolds with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as holonomy group and 3-step nilpotent universal covering group are all non-orientable and so we did not have to make any calculations.

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$				
Family	k	Orientable		Non-orientable
		An.th. holds	An.th. holds not	An.th. holds not
7[Q=(2ℓ)] (-1,1)	$k \equiv 0 \bmod 2$	0	0	1
7[Q=(2ℓ+1)] (-1,1)	/	0	0	1
8 (-1,1)	/	0	0	1

To conclude this section, we can say that this section further motivates to investigate the validity of the Anosov theorem for infra-nilmanifolds with filiform universal covering group.

Part IV

Nederlandstalige samenvatting

Hoofdstuk 9

De Anosov relatie voor infra-nilvariëteiten

In deze thesis onderzoeken we de 'Anosov relatie' voor continue afbeeldingen f van een infra-nilvariëteit naar zichzelf. We zeggen dat f aan de Anosov relatie voldoet als het Nielsen getal $N(f)$ en het Lefschetz getal $L(f)$ op teken na gelijk zijn: $N(f) = |L(f)|$. Deze aan f geassocieerde getallen vinden hun oorsprong in de vastepuntstheorie. Het Nielsen getal geeft ons het meeste informatie maar is moeilijk te berekenen vertrekkende uit de definitie, terwijl voor het Lefschetz getal het omgekeerde geldt. Daarom proberen we de twee getallen aan elkaar te linken en D. Anosov bewees in 1985 de volgende stelling (zie [1]).

Stelling 9.1. *Voor elke continue afbeelding $f : N \to N$ van een nilvariëteit naar zichzelf, geldt dat $N(f) = |L(f)|$.*

In dit werk proberen we deze stelling te veralgemenen en we doen dit op twee manieren. Een eerste manier is de geldigheid van deze stelling aan te tonen voor andere klassen van variëteiten dan de nilvariëteiten zoals bijvoorbeeld reeds gedaan werd door E. Keppelmann en C. McCord voor exponentiële solvvariëteiten (zie [31]). Wij doen in deze thesis hetzelfde voor klassen van infra-nilvariëteiten.

Een tweede manier is nagaan of de stelling van Anosov niet geldig is voor bepaalde types van afbeeldingen op een gegeven variëteit. Zo toonden S. Kwasik en K.B. Lee in [34] aan dat het Anosov theorema altijd geldt voor homotoop periodische afbeeldingen van infra-nilvariëteiten en W. Malfait breidde dit uit naar virtueel unipotente afbeeldingen zie ([42]). Ook wij bekeken de geldigheid van het Anosov theorema voor enkele types van afbeeldingen.

Naast het aantonen van de theoretische resultaten gaan we ook na wat we al weten, of misschien interessanter nog niet weten, over de geldigheid van het Anosov theorema voor infra-nilvariëteiten in lage dimensies, namelijk tot en met dimensie 4.

Hierna vatten we het Engelstalige gedeelte van deze thesis kort samen. We opteerden hierbij om per hoofdstuk een nieuwe sectie te voorzien. We hopen dat de geïnteresseerde lezer zo gemakkelijker zijn weg vindt naar de meer gedetailleerde passages en de nodige bewijzen die we hieronder niet herhaalden.

9.1 Continue afbeeldingen van infra-nilvariëteiten

In deze thesis bestuderen we steeds dezelfde klasse van variëteiten, namelijk de infra-nilvariëteiten en om deze te definiëren beginnen we met enkele notaties in te voeren.

Als we met een Lie groep G werken dan veronderstellen we steeds dat G een samenhangende, enkelvoudig samenhangende, nilpotente Lie groep is. Een affien endomorfisme van G is een element (g, φ) van de semigroep $G \rtimes \mathrm{Endo}(G)$ waarbij g het translatiegedeelte en φ het lineaire gedeelte is. Het product van twee affiene endomorfismen wordt gegeven door $(g, \varphi)(h, \mu) = (g \cdot \varphi(h), \varphi\mu)$. In het speciale geval dat $\varphi \in \mathrm{Aut}(G)$, is (g, φ) een inverteerbaar affiene transformatie van G en we duiden dit aan met $(g, \varphi) \in \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$.

Een groep E is een bijna-kristallografische groep als het een deelgroep is van $\mathrm{Aff}(G)$ zodat de deelgroep van pure translaties $N = E \cap G$, een uniform rooster is in G en van eindige index is in E. We noemen M een infra-nilvariëteit als zijn eerste fundamentaalgroep $\pi_1(M)$ een torsievrije bijna-kristallografische, of kortweg bijna-Bieberbach, groep is. Omdat N van eindige index is in E, is $F = E/N$ een eindige groep en deze wordt de holonomiegroep van M genoemd. Het is op basis van deze F dat wij onze klassen van infra-nilvariëteiten bepalen.

Elke bijna-kristallografische groep bepaalt een trouwe representatie $T : F \to \mathrm{Aut}(G)$ die we de holonomierepresentatie noemen. Door hiervan de differentiaal te nemen, bekomen we dan tevens een trouwe representatie

$$T_* : F \to \mathrm{Aut}(\mathfrak{g}) : x \mapsto T_*(x) = d(T(x))$$

waarbij $\mathfrak{g}$ de Lie algebra horende bij G is. De holonomierepresentatie is cruciaal voor onze resultaten en een eerste toepassing is volgende propositie.

Propositie 9.2. *Zij M een infra-nilvariëteit met holonomiegroep F en holonomierepresentatie $T : F \to \mathrm{Aut}(G)$. Dan geldt*

- *M is oriënteerbaar $\Leftrightarrow \forall x \in F : \det(T_*(x)) = 1$;*
- *M is niet oriënteerbaar $\Leftrightarrow \exists x \in F : \det(T_*(x)) = -1$.*

In het speciale geval dat G abels is, met andere woorden $G = \mathbb{R}^n$, noemen we G een platte variëteit. In dit geval is de groep van pure translaties altijd isomorf met $\mathbb{Z}^n$ waarbij n de dimensie van de variëteit is. Bovendien is in dit geval $T = T_*$ en kan dit gezien worden als een afbeelding naar $\mathrm{Gl}(n, \mathbb{Z})$. Dit alles impliceert dat de notaties en berekeningen vergemakkelijkt worden in het geval we werken met platte variëteiten.

De korte inleiding hierboven geeft aan dat we de infra-nilvariëteiten algebraïsch kunnen beschrijven. Inderdaad, aan elke infra-nilvariëteit kunnen we een unieke bijna-Bieberbach groep associëren (en omgekeerd geldt hetzelfde). Concreet wordt dit alles als volgt samengevat: zij M een infra-nilvariëteit met bijna-Bieberbach groep E, dan is $M = E\backslash G$. Dankzij K.B. Lee hebben we ook dergelijke algebraïsche beschrijving voor continue afbeeldingen van infra-nilvariëteiten (zie [38]). Hij bewees namelijk volgende stelling met het voor ons belangrijk bijhorend gevolg.

Stelling 9.3. *Zij $E, E' \subset \mathrm{Aff}(G)$ twee bijna-kristallografische groepen. Dan geldt voor elk homomorfisme $\theta : E \to E'$, dat er een $g = (\delta, \mathfrak{D}) \in G \rtimes \mathrm{Endo}(G)$ bestaat, zodat $\theta(\alpha) \cdot g = g \cdot \alpha$ voor elke $\alpha \in E$.*

Gevolg 9.4. *Zij $M = E\backslash G$ een infra-nilvariëteit en $f : M \to M$ een continue afbeelding. Dan is f homotoop met een afbeelding $h : M \to M$ geïnduceerd door een affien endomorfisme $(\delta, \mathfrak{D}) : G \to G$.*

Dit affien endomorfisme wordt ook een homotopie lift genoemd en van een gegeven afbeelding f kunnen we een homotopie lift vinden, door gebruik te maken van het feit dat f een homomorfisme $f_* : \pi_1(M) \to \pi_1(M)$ induceert. Merk bovendien op dat deze resultaten inderdaad, op homotopie na, een algebraïsche beschrijving geven van alle continue afbeeldingen van een gegeven infra-nilvariëteit M. Construeer namelijk alle mogelijke homomorfismen van de fundamentaalgroep (dit is een eindig proces omdat deze groep eindig voortgebracht is) en bereken zo alle mogelijke affiene endomorfismen. Elke continue afbeelding van M is dan homotoop met een afbeelding geïnduceerd door één van deze affiene endomorfismen.

We leiden van deze stelling nog een tweede gevolg af.

Gevolg 9.5. *Zij $M = E\backslash G$ een infra-nilvariëteit met holonomiegroep F en holonomierepresentatie $T : F \to \mathrm{Aut}(G)$. Veronderstel dat $f : M \to M$ een continue afbeelding is en $(\delta, \mathfrak{D})$ een homotopie lift is van f. Dan geldt*

$$\forall x \in F, \exists y \in F : T_*(y)\mathfrak{D}_* = \mathfrak{D}_* T_*(x).$$

9.2 De Anosov relatie

In de voorgaande sectie gaven we kort aan dat we het topologisch onderwerp van deze thesis, namelijk continue afbeeldingen van infra-nilvariëteiten, volledig algebraïsch kunnen beschrijven. Uiteraard is dit voor ons enkel nuttig als we de Anosov relatie ook algebraïsch kunnen onderzoeken. Dit is inderdaad mogelijk en de nodige resultaten hieromtrent werden opnieuw in [38] door K.B. Lee gegeven. Maar vooraleer deze resultaten te geven, introduceren we kort het Nielsen getal $N(f)$ en het Lefschetz getal $L(f)$.

Zij $f : X \to X$ een continue afbeelding van een variëteit X naar zichzelf en zij $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$ de verzameling van vaste punten van f. Deze verzameling wordt opgedeeld in equivalentieklassen, die we vastepuntsklassen noemen, door de volgende relatie: $x, y \in \mathrm{Fix}(f)$ behoren tot dezelfde klasse als en slechts als er een pad w van x naar y bestaat zodat w en fw homotoop zijn.
Sommige vastepuntsklassen kunnen verwijderd worden door een homotopie. Dit betekent dat er een homotope afbeelding bestaat waarvoor de punten uit de oorspronkelijke vastepuntsklasse niet langer vaste punten zijn. Dergelijke vastepuntsklassen worden niet essentieel genoemd en vastepuntsklassen die niet verwijderd kunnen worden door een homotopie worden dan natuurlijk essentieel genoemd. Omdat in de studie van de vaste punten men enkel geïnteresseerd is in de essentiële vastepuntsklassen, wordt $N(f)$ als volgt gedefinieerd.

Definitie 9.6. *Zij $f : X \to X$ een continue afbeelding van een variëteit X naar zichzelf, dan is $N(f)$ gelijk aan het aantal essentiële vastepuntsklassen van f.*

Zonder hier dieper om in te gaan, blijkt wel dat $N(f)$ moeilijk te berekenen is vanuit zijn definitie. Om $N(f)$ te berekenen moeten we in principe namelijk eerst alle afbeeldingen homotoop met f construeren, wat niet zo evident is.

Het Lefschetz getal wordt als volgt gedefinieerd.

Definitie 9.7. *Zij $f : X \to X$ een continue afbeelding op een compact samenhangende variëteit X. Dan is*

$$L(f) = \sum_i (-1)^i \ \mathrm{Trace}(f_* : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})).$$

Het Lefschetz getal is veel beter berekenbaar en geeft ons beperkte informatie over de vaste punten van f. Namelijk als $L(f) \neq 0$ dan heeft f minstens één vast punt, maar over het precieze aantal hebben we geen informatie. Een tweede, veel krachtiger, toepassing is natuurlijk Stelling 9.1 die wij proberen te veralgemenen. D. Anosov zelf echter toonde reeds aan dat zijn resultaat niet geldig is voor alle infra-nilvariëteiten want hij construeerde een tegenvoorbeeld op de fles van Klein die een 2-dimensionale platte variëteit met holonomiegroep $\mathbb{Z}_2$ is.

Omdat de veralgemening van Stelling 9.1 het onderwerp is van deze thesis, geven we nog eens volgende formele definitie.

Definitie 9.8. *Zij M een infra-nilvariëteit.*

- *Een continue afbeelding $f : M \to M$ voldoet aan de Anosov relatie als en slechts als $N(f) = |L(f)|$.*
- *M voldoet aan het Anosov theorema als en slechts als elke continue afbeelding $f : M \to M$ voldoet aan de Anosov relatie.*

Nu kunnen we de algebraïsche vertaling van ons topologisch probleem vervolledigen wegens de volgende stelling van K.B. Lee (zie [38]).

Stelling 9.9. *Zij $f : M \to M$ een continue afbeelding van een infra-nilvariëteit M naar zichzelf en zij $T : F \to \mathrm{Aut}(G)$ de geassocieerde holonomierepresentatie. Als $(\delta, \mathfrak{D}) \in G \rtimes \mathrm{Endo}(G)$ een homotopie lift is van f, dan geldt*

- *$N(f) = L(f)$ als en slecht als $\forall x \in F : \det(I_n - T_*(x)\mathfrak{D}_*) \geq 0$.*
- *$N(f) = -L(f)$ als en slechts als $\forall x \in F : \det(I_n - T_*(x)\mathfrak{D}_*) \leq 0$.*

Deze stelling is cruciaal voor de bewijzen van onze resultaten en we kunnen ze al een eerste keer gebruiken om de volgende propositie aan te tonen.

Propositie 9.10. *Zij M een infra-nilvariëteit met holonomiegroep F en geassocieerde holonomierepresentatie $T : F \to \mathrm{Aut}(G)$. Zij $f : M \to$*

M een continue afbeelding met homotopie lift $(\delta, \mathfrak{D})$. Veronderstel dat $\forall x \in F,\ x \neq 1:\ T_*(x)\mathfrak{D}_* \neq \mathfrak{D}_* T_*(x)$, dan geldt

$$\forall x \in F : \det(I_n - \mathfrak{D}_*) = \det(I_n - T_*(x)\mathfrak{D}_*)$$

en dus $N(f) = |L(f)|$.

9.3 De periodische rij

Voor alle infra-nilvariëteiten met holonomiegroep van oneven orde geldt het Anosov theorema. Om dit aan te tonen introduceren we de periodische rij geassocieerd aan een afbeelding en deze techniek kunnen we ook gebruiken om de Anosov relatie te onderzoeken van de expanderende en de nergens expanderende afbeeldingen.

Zij $f : M \to M$ een continue afbeelding van een infra-nilvariëteit naar zichzelf, $(\delta, \mathfrak{D})$ een homotopie lift van f en $T : F \to \mathrm{Aut}(G)$ de geassocieerde holonomierepresentatie. Gebruik makend van Gevolg 9.5 kunnen we voor elke $x_1 \in F$ een rij $x_1, x_2, \ldots$ creëren zodat $\forall i \in \mathbb{N}_0$: $\mathfrak{D}_* T_*(x_i) = T_*(x_{i+1})\mathfrak{D}_*$. Aangezien F eindig is, weten we dat we deze rij zodanig kunnen construeren dat ze op een gegeven moment periodisch wordt. Namelijk, er moet een $j \geq 1$ en $k \geq 1$ bestaan zodat $x_{j+k} = x_j$ en bijgevolg kunnen we er voor zorgen dat $x_{j+k+1} = x_{j+1}$, $x_{j+k+2} = x_{j+2}, \ldots$.
We noemen dergelijke rij **een periodische rij voor x_1, geassocieerd aan f, met periode k beginnend van positie j**. Voor deze rijen hebben we het volgende lemma.

Lemma 9.11. *Zij M een infra-nilvariëteit met holonomiegroep F en geassocieerde holonomierepresentatie $T : F \to \mathrm{Aut}(G)$. Zij $f : M \to M$ een continue afbeelding en $(\delta, \mathfrak{D})$ een homotopie lift van f. Veronderstel dat $x_1 \in F$ en $x_1, x_2, x_3, \ldots$ een periodische rij is voor x_1 geassocieerd aan f, met periode k beginnend van positie j. Dan geldt*

1. *$\forall i \in \mathbb{N}_0$: $\det(I_n - T_*(x_1)\mathfrak{D}_*) = \det(I_n - T_*(x_i)\mathfrak{D}_*)$;*
2. *$\mathfrak{D}_*^k T_*(x_j) = T_*(x_j)\mathfrak{D}_*^k$;*
3. *$\exists l \in \mathbb{N}_0$: $(T_*(x_j)\mathfrak{D}_*)^l = \mathfrak{D}_*^l$.*

Met dit lemma en steunend op Stelling 9.9 kunnen we dan volgende stelling bewijzen.

Stelling 9.12. *Als M een infra-nilvariëteit is met holonomiegroep F van oneven orde, dan is $N(f) = |L(f)|$ voor elke continue afbeelding $f : M \to M$.*

Een cruciaal gegeven in het bewijs is dat de matrices $T_*(x)$ (met $T : F \to \mathrm{Aut}(G)$ de geassocieerde holonomierepresentatie en $x \in F$) niet -1 als eigenwaarde kunnen hebben omdat F van oneven orde is. Merk op dat dit namelijk impliceert dat al deze matrices ook van eindige, oneven orde zijn. Als F van even orde is, dan is -1 natuurlijk wel een eigenwaarde en dit zorgt voor complicaties die we in de volgende secties telkens op andere manieren proberen op te lossen.

In deze sectie doen we dit door te kijken naar 2 types van afbeeldingen: de welgekende expanderende afbeeldingen en de door ons gedefinieerde nergens expanderende afbeeldingen. Veronderstel dat $f : M \to M$ een continue afbeelding is van een infra-nilvariëteit naar zichzelf en $(\delta, \mathfrak{D})$ een homotopie lift is van f. Kort samengevat mogen we stellen dat f een expanderende afbeelding op een infra-nilvariëteit is als en slechts als voor elke eigenwaarde λ van $\mathfrak{D}_*$ geldt dat $|\lambda| > 1$. Zoals wellicht verwacht geldt voor een nergens expanderende afbeelding precies het omgekeerde: voor elke eigenwaarde λ van $\mathfrak{D}_*$ geldt dan dat $|\lambda| \leq 1$.

Voor de expanderende afbeeldingen kunnen we, met behulp van de periodische rij, het volgende aantonen.

Stelling 9.13. *Zij $f : M \to M$ een expanderende afbeelding van een infra-nilvariëteit naar zichzelf, dan is $N(f) = |L(f)|$ als en slechts als M oriënteerbaar is.*

Dit resultaat kunnen we nu herformuleren om de geldigheid van het Anosov theorema na te gaan voor niet-oriënteerbare infra-nilvariëteiten. Namelijk van zodra deze een expanderende afbeelding toelaten, mogen we concluderen dat het Anosov theorema niet geldig is. Op platte variëteiten en 2-staps nilpotente infra-nilvariëteiten bestaan er altijd expanderende afbeeldingen (zie [22] en [36]), maar dit is niet zo in het algemeen. Merk op dat we een infra-nilvariëteit k-staps nilpotent noemen als de bijhorende Lie groep G k-staps nilpotent is. We verwijzen naar Sectie 3.3 waarin we een niet-oriënteerbare infra-nilvariëteit M met $\mathbb{Z}_2$ als holonomiegroep presenteren waarvoor het Anosov theorema geldig is (en waarvoor M dus geen expanderende afbeeldingen toelaat).

Propositie 9.14. *Als M een niet oriënteerbare platte variëteit of een niet oriënteerbare 2-staps nilpotente infra-nilvariëteit is, dan geldt het Anosov theorema niet.*

Voor de nergens expanderende afbeeldingen hebben we een nog sterker resultaat (dat we opnieuw bekomen met behulp van de periodische rij).

Stelling 9.15. *Zij $f : M \to M$ een nergens expanderende afbeelding van een infra-nilvariëteit naar zichzelf, dan geldt dat $N(f) = L(f)$.*

9.4 Anosov diffeomorfismen

In deze sectie onderzoeken we een derde type van afbeeldingen, namelijk de Anosov diffeomorfismen. Opnieuw kort samengevat kunnen we stellen dat een diffeomorfisme $f : M \to M$ van een platte variëteit M naar zichzelf en met homotopie lift $(\delta, \mathfrak{D})$ een Anosov diffeomorfisme is als $\mathfrak{D}_*$ geen eigenwaarden van modulus 1 heeft. Deze definitie kan exacter geformuleerd worden voor variëteiten in het algemeen. Zoals bij de expanderende afbeeldingen, laten niet alle infra-nilvariëteiten Anosov diffeomorfismen toe. Voor de platte variëteiten hebben we echter dankzij H. Porteous een complete omschrijving van de variëteiten die Anosov diffeomorfismen toelaten (zie [49]).

Stelling 9.16. *Een n-dimensionele platte variëteit met geassocieerde holonomierepresentatie $T : F \to \mathrm{Gl}(n, \mathbb{Z})$ laat een Anosov diffeomorfisme toe als en slecht als elke $\mathbb{Q}$-irreduciebele component van T van multipliciteit 1 reduciebel is over $\mathbb{R}$.*

Bovendien argumenteert J. Lauret dat een dergelijke beschrijving niet kan bestaan voor de infra-nilvariëteiten in het algemeen (zie [35]). Daarom beperken we onszelf tot de platte variëteiten.

In [41] werd bewezen dat als een platte variëteit M een Anosov diffeomorfisme toelaat, het eerste Betti getal $b_1(M)$ dan voldoet aan één van volgende voorwaarden: $b_1(M) = 0, 2 \leq b_1(M) \leq n - 2$ of $b_1(M) = n$. Het eerste Betti getal is per definitie gelijk aan de rang van de eerste homologie groep $H_1(M, \mathbb{Z})$. Als $b_1(M) = n$ dan is M de torus en weten we dus al dat het Anosov theorema geldt. Voor de andere twee gevallen hebben we geconstateerd dat, op enkele randgevallen na, het feit dat we werken met Anosov diffeomorfismen niet erg restrictief is met betrekking tot de Anosov relatie van deze afbeeldingen. Onze resultaten zijn samengevat in de volgende tabel die tevens de verwijzingen bevat naar de respectievelijk theorema's in Hoofdstuk 4. Hierbij is f steeds een Anosov diffeomorfisme op een platte variëteiten M in de respectievelijke dimensie n en Betti getal $b_1(M)$. We geven steeds aan of er al dan niet een voorbeeld bestaat voor de relatie op de eerste lijn.

$b_1(M)$	n	$N(f) = \|L(f)\|$	$N(f) \neq \|L(f)\|$	Proof
$b_1(M) = 0$	$n = 6$	altijd	nooit	Theorema 4.16
$b_1(M) = 0$	$n > 6$	bestaat	bestaat	Theorema 4.14
$2 \leq b_1(M) < n - 2$	$n > 4$	bestaat	bestaat	Theorema 4.6
$b_1(M) = n - 2$	$n \geq 4$	nooit	altijd	Theorema 4.10
$b_1(M) = n$	$n \geq 1$	altijd	nooit	zie [1]

9.5 Infra-nilvariëteiten met cyclische holonomiegroep

In deze sectie beperken we ons tot infra-nilvariëteiten met cyclische holonomiegroep. Uit Stelling 9.12 weten we reeds dat het Anosov theorema geldt voor een groot gedeelte van deze infra-nilvariëteiten. Echter wegens het tegenvoorbeeld van D. Anosov op de fles van Klein (met $\mathbb{Z}_2$ als holonomiegroep) weten we ook dat het Anosov theorema niet geldig is voor alle infra-nilvariëteiten van deze klasse. Bijgevolg hebben we zeker nog een extra voorwaarde nodig.

Stelling 9.17. *M is een infra-nilvariëteit met cyclische holonomiegroep F voortgebracht door x_0. Zij $T : F \to \mathrm{Aut}(G)$ de geassocieerde holonomierepresentatie en veronderstel dat -1 geen eigenwaarde is van $T_*(x_0)$. Dan geldt voor elke continue afbeelding $f : M \to M$ dat $N(f) = |L(f)|$.*

Merk op dat de extra voorwaarde opnieuw over de eigenwaarde -1 gaat, maar nu kunnen we die niet volledig uitsluiten zoals in Sectie 9.3 het geval was. Als we immers machten van $T_*(x_0)$ nemen dan kan -1 wel een eigenwaarde zijn, wat het bewijzen van bovengaande stelling veel complexer maakt.

Veronderstel dat de orde van F gelijk is aan $2^r k$ met $r \geq 0$ en k een oneven positief geheel getal. Zij dan $d_0, d_1, \ldots, d_t$ de delers van $2^r k$ zodat $1 = d_0 < d_1 < \cdots < d_t = 2^r k$. Het bewijs van stelling 9.17 is gebaseerd op het feit dat er een matrix $P \in \mathrm{Gl}(n, \mathbb{R})$ bestaat waarmee we $T_*(x_0)$ als volgt kunnen herschrijven:

$$PT_*(x_0)P^{-1} = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & A_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_t \end{pmatrix}.$$

Hierbij zijn de $A_i \in \mathrm{Gl}(n_i, \mathbb{R})$, $n_1 + \cdots n_t = n$ en de A_i hebben enkel eigenwaarden van orde d_i. Met dezelfde P kunnen we dan voor alle geschikte affiene endomorfismen $(\delta, \mathfrak{D})$ de $\mathfrak{D}_*$ schrijven als

$$P\mathfrak{D}_*P^{-1} = \begin{pmatrix} D_0 & 0 & \cdots & 0 \\ * & D_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & D_t \end{pmatrix}$$

met $D_i \in M_{n_i}(\mathbb{R})$. Bijgevolg bekomen we dat de determinanten van Stelling 9.9 allen van de volgende vorm zijn

$$\begin{aligned} \det(I_n - T_*(x_0^m)\mathfrak{D}_*) &= \det(I_n - PT_*(x_0^m)P^{-1}P\mathfrak{D}_*P^{-1}) \\ &= \det(I_{n_0} - A_0^m D_0) \cdots \det(I_{n_t} - A_t^m D_t) \end{aligned}$$

en het volstaat dus om de tekens van de factoren afzonderlijk te bepalen (gebruik makend van het feit dat de A_i enkel eigenwaarden van orde d_i hebben). Dit wordt gedaan in verschillende lemma's die samen uiteindelijk leiden tot het bewijs van bovengaande stelling.

Stelling 9.17 geeft ons enkel een noodzakelijke voorwaarde opdat het Anosov theorema geldt. De volgende vraag is dan natuurlijk wat met het Anosov theorema als de extra voorwaarde niet voldaan is? Voor platte variëteiten kunnen we altijd een tegenvoorbeeld construeren van zodra deze voorwaarde niet voldaan is.

Propositie 9.18. *Stel dat M een n-dimensionele, oriënteerbare, platte variëteit met cyclische holonomiegroep F voortgebracht door x_0 is. Zij $T : F \to \mathrm{Gl}(n, \mathbb{Z})$ de geassocieerde holonomierepresentatie en veronderstel dat -1 wel een eigenwaarde is van $T_*(x_0)$.*
Dan bestaat er altijd een continue afbeelding $f : M \to M$ zodat $N(f) \neq |L(f)|$.

Voor infra-nilvariëteiten in het algemeen is dit niet het geval en hiervoor kunnen we hetzelfde voorbeeld gebruiken als in Sectie 3.3 want de holonomiegroep is $\mathbb{Z}_2$.

9.6 De platte veralgemeende Hantzsche-Wendt variëteiten

De originele Hantzsche-Wendt variëteit is een 3-dimensionele, platte variëteit met $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ als holonomiegroep. We noemen daarom een n-dimensionele, platte variëteit M een veralgemeende Hantzsche-Wendt variëteit als zijn holonomiegroep $\mathbb{Z}_2^{n-1}$ is.

Omdat we met dergelijke holonomiegroep werken, zijn al de matrices geassocieerd aan de holonomierepresentatie, op de eenheidsmatrix na, van orde 2. Dit betekent dat we in een tegenovergestelde situatie zitten in vergelijking met de vorige secties waarin we meestal de 'afwezigheid' van de eigenwaarde -1 verkozen. Desalniettemin kunnen we toch aantonen dat voor de platte oriënteerbare veralgemeende Hantzsche-Wendt variëteiten het Anosov Theorema steeds geldt. Merk op dat n oneven moet zijn omdat er geen platte oriënteerbare veralgemeende Hantzsche-Wendt variëteiten bestaan als n even is (zie gevolg 6.2).

Stelling 9.19. *Zij $n \geq 3$ oneven en M een platte, oriënteerbare, n-dimensionele, veralgemeende Hantzsche-Wendt variëteit. Dan geldt voor elke continue afbeelding $f : M \to M$ dat $N(f) = |L(f)|$.*

Zonder in detail te willen treden is het belangrijk iets op te merken over het bewijs van deze stelling. De redenering van het bewijs is volledig analoog aan die in bewijzen van de vorige stellingen. Namelijk we willen Stelling 9.9 toepassen wat impliceert dat we tekens van determinanten moeten bepalen en om dit te doen gebruiken we de eigenschappen van de holonomiegroep. In tegenstelling tot de vorige bewijzen is dit hier echter niet genoeg en moeten we ook de translatiegedeelten van de generatoren van de fundamentaalgroep expliciet gebruiken.

9.7 Het Anosov theorema in lage dimensies

In het derde deel van deze thesis gaan we na wat we weten over het Anosov theorema voor infra-nilvariëteiten in lage dimensies, i.e. tot en met dimensie 4. Zo tonen we onder andere aan dat reeds veel variëteiten behandeld worden door onze resultaten. Maar misschien nog veel interessanter is het feit dat we zo ook variëteiten ontdekken die kunnen leiden tot nieuwe onderzoeksvragen.

Voor alle variëteiten waarvoor het Anosov theorema niet geldt wordt een tegenvoorbeeld gegeven en voor de andere variëteiten voorzien we de nodige argumenten waarom het Anosov theorema opgaat.

In Hoofdstuk 7 behandelen we de platte variëteiten. Om dit gestructureerd te doen, hebben we een omschrijving nodig van alle mogelijke variëteiten of, gegeven het unieke verband, een omschrijving van alle Bieberbach groepen. Hiervoor gebruikten we [7]. Omwille van de voor de hand liggende redenen gaan we hier niet alle resultaten herhalen. Maar kort samengevat kunnen we stellen dat in dimensie 1 en 2 Anosov reeds alles behandelde aangezien we daar enkel de torus en de fles van Klein hebben. In dimensie 3 worden alle 10 de platte variëteiten reeds behandeld door de hiervoor vermelde resultaten en in dimensie 4 zijn 55 van de 74 variëteiten behandeld.

Van het onderzoek van de 19 overige variëteiten willen we twee conclusies vermelden. Uit de resultaten van de 10 variëteiten met $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ als holonomiegroep leiden we onderstaande propositie met betrekking tot $\mathbb{Z}$-klassen af. Merk op dat twee Bieberbach groepen tot dezelfde $\mathbb{Z}$-klasse behoren als de matrices van de geassocieerde holonomierepresentaties geconjugeerd zijn over $\mathbb{Z}$.

Propositie 9.20. *Er bestaan platte variëteiten M_1 en M_2 waarvan de fundamentaalgroepen tot dezelfde $\mathbb{Z}$-klasse behoren zodat het Anosov theorema geldig is voor M_1 en niet voor M_2.*

Van de 9 variëteiten met niet-abelse holonomiegroep onthouden we vooral de onderstaande vraag. Deze komt voort uit de vaststelling dat in dimensie 4, het Anosov theorema steeds geldt voor de infra-nilvariëteiten met A_4 als holonomiegroep.
Voor de andere infra-nilvariëteiten met niet-abelse holonomiegroep deden we gelijkaardige vaststelling als bij de infra-nilvariëteiten met abelse holonomiegroep.

Vraag 9.21. *Geldt het Anosov theorema voor elke (platte) variëteit met holonomiegroep A_4?*

Voor de infra-nilvariëteiten, waarbij G niet abels is, moesten we in principe hetzelfde doen en nu gebruikten we [12] als omschrijving van alle bijna-Bieberbach groepen. Er zijn echter wat complicaties omdat de matrices niet zomaar afgeleid kunnen worden van de presentaties van de groepen. Bovendien hangen alle groepen af van ten hoogste 7 parameters die op hun beurt aan voorwaarden moeten voldoen. Dit zorgt ervoor dat we niet zomaar alles kunnen samenvatten zoals hierboven.

We onthouden van deze berekeningen vooral dat de structuur van de bijhorende nilpotente Lie groep G een grote invloed kan hebben op de geldigheid van het Anosov theorema. Deze structuur zorgt immers voor extra restricties op de lineaire gedeelten van de geschikte affiene endomorfismen $(\delta, \mathfrak{D})$. De invloed van deze restricties is maximaal als G een filiforme Lie groep is, i.e. een n-dimensionele, $(n-1)$-staps nilpotente Lie groep. Dit alles leidt ons naar de volgende vragen.

Vraag 9.22. *Wat kunnen we zeggen over het Anosov theorema voor de infra-nilvariëteiten waarvan de bijhorende Lie groep filiform is?*
Of meer algemeen: kunnen we een verband vinden tussen de structuur van de bijhorende Lie groep en de geldigheid van het Anosov theorema?

Wat de eerste vraag betreft, in dimensie 4 geldt het Anosov theorema steeds voor de oriënteerbare 3-staps nilpotente infra-nilvariëteiten. Daarentegen met betrekking tot de tweede vraag, geven we ook voorbeelden die duidelijk illustreren dat het verband met de structuur zeer delicaat kan zijn.

Voor de infra-nilvariëteiten met niet-abelse holonomiegroep bekomen we opnieuw interessante resultaten, maar deze keer voor alle variëteiten. We constateren namelijk voor alle 4-dimensionele infra-nilvariëteiten met niet-abelse holonomiegroep dat het Anosov theorema steeds geldig is. In de zoektocht naar het beantwoorden van bovenstaande vragen, kunnen we dus deze klasse van infra-nilvariëteiten misschien apart beschouwen.

References

1. Anosov, D. *The Nielsen numbers of maps of nil-manifolds.* Uspekhi. Mat. Nauk, 1985, 40, pp. 133–134. English transl.: Russian Math. Surveys, 40 (no. 4), 1985, pp. 149–150.
2. Brooks, R., Brown, R. F., Pak, J., and Taylor, D. *The Nielsen number of maps on tori.* Proc. Amer. Math. Soc., 1975, 52, pp. 398–400.
3. Brown, H., Bülow, R., Neubüser, J., Wondratscheck, H., and Zassenhaus, H. *Crystallographic groups of four–dimensional Space.* Wiley New York, 1978.
4. Brown, K. S. *Cohomology of groups.*, volume 87 of *Grad. Texts in Math.* Springer-Verlag New York Inc., 1982.
5. Brown, R. F. *The Lefschetz fixed point theorem.* Scott, Foresman and Company, 1971.
6. Brown, R. F. *Nielsen Fixed point theory on manifolds.* Banach Cent. Publ., 1999, 49, pp. 19–27.
7. *CARAT – Crystallographic AlgoRithms And Tables, Version 1.2.* Lehrstuhl B fur Mathematik, Aachen, 2001. (`http://wwwb.math.rwth-aachen.de/carat/`).
8. Charlap, L. S. *Bieberbach Groups and Flat Manifolds.* Universitext. Springer–Verlag, New York Inc., 1986.
9. Cid, C. and Schulz, T. *Computation of five- and six-dimensional Bieberbach groups.* Exp. Math., 2001, 10, 1, pp. 109–115.
10. Cobb, P. V. *Manifolds with holonomy group* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and first betti number zero.* J. Differential Geom., 1975, 10, 221–224.
11. Dekimpe, K. *The construction of affine structures on virtually nilpotent groups.* Manuscripta Math., 1995, 87 pp. 71–88.
12. Dekimpe, K. *Almost-Bieberbach Groups: Affine and Polynomial Structures*, volume 1639 of *Lect. Notes in Math.* Springer–Verlag, 1996.
13. Dekimpe, K., De Rock, B., and Malfait, W. *The Anosov relation for Nielsen numbers of maps of infra-nilmanifolds.* To appear in Monatschefte für Mathematik.
14. Dekimpe, K., De Rock, B., and Malfait, W. *The Anosov theorem for infra-nilmanifolds with cyclic holonomy group.* To appear in Pacific Journal of Mathematics.
15. Dekimpe, K., De Rock, B., and Malfait, W. *The Anosov theorem for flat generalized Hantzsche-Wendt manifolds.* J. Geometry and physics, 2004, 52 pp. 174–185.

16. Dekimpe, K., De Rock, B., and Malfait, W. *The Nielsen numbers of Anosov diffeomorphisms on flat Riemannian manifolds.* Forum Mathematicum, 2005, 17 pp. 325–341.
17. Dekimpe, K., De Rock, B., and Pouseele, H. *The Anosov theorem for infra-nilmanifolds with an odd order abelian holonomy group.* To appear in Fixed Point Theory and Applications.
18. Dekimpe, K. and Eick, B. *Computational aspects of Group extensions and their applications to topology.* Experimental Math., 2002, 11 pp. 183–200.
19. Dekimpe, K. and Lee, K. B. *Expanding maps on infra-nilmanifolds of homogeneous type.* Trans. Amer. Math. Soc., 2003, 355 (3), pp. 1067–1077.
20. Dixmier, J. and Lister, W. *Derivations of nilpotent Lie algebras.* Proc. Am. Math. Soc., 1957, 8 155–158.
21. Dugundji, J. and Granas, A. *Fixed point theory.* Springer monographs in mathematics. Springer-Verlag, New York, 2003.
22. Epstein, D. and Shub, M. *Expanding endomorphisms of flat manifolds.* Topology, 1968, 7 pp. 139–141.
23. Franks, J. *Anosov diffeomorphisms.* Global Analysis: Proceedings of the Symposia in Pure Mathematics, 1970, 14, pp. 61–93.
24. Goze, M. and Khakimdjanov, Y. *Nilpotent Lie Algebras.* Mathematics and Its Applications. Kluwer Academic Publishers, 1996.
25. Gromov, M. *Groups of polynomial growth and expanding maps.* Institut des Hautes Études Scientifiques, 1981, 53, pp. 53–73.
26. Hantzsche, W. and Wendt, H. *Driedimensionale euklidische Raumformen.* Math. Ann., 1935, 110, pp. 593–611.
27. Heath, P. R. and Keppelmann, E. C. *Model solvmanifolds for Lefschetz and Nielsen theories.* Quaest. Math., 2002, 25 4, pp. 483–501.
28. Jezierski, J., Kędra, J., and Marzantowicz. *Homotopy minimal periods for NR-solvmanifold maps.* Topology and applications, 2004, no.1-3, 29–49.
29. Jiang, B. *Nielsen Fixed Point Theory*, volume 14 of *Contemporary Math.* American Mathematical Society, 1983.
30. Jiang, B. *Commutativity and Wecken properties for fixed points of surfaces and 3-manifolds.* Topology Applications, 1993, 53 53, pp. 221–228.
31. Keppelmann, E. C. and McCord, C. K. *The Anosov theorem for exponential solvmanifolds.* Pacific J. Math., 1995, 170, No. 1, pp. 143–159.
32. Kiang, T.-h. *The Theory of Fixed Point Classes.* Springer-Verlag, 1989.
33. Kim, J. B., Seung Won; Lee and Lee, K. B. *Averaging formula for Nielsen numbers.* Nagoya Math. J., 2005, 178,.
34. Kwasik, S. and Lee, K. B. *The Nielsen numbers of homotopically periodic maps of infra-nilmanifolds.* J. London Math. Soc. (2), 1988, 38, pp. 544–554.
35. Lauret, J. *Examples of Anosov diffeomorphisms.* J. Algebra, 2003, 262 1, 201–209.
36. Lee, H. and Lee, K. B. *Expanding maps on 2-step infra-nilmanifolds.* Topology Appl., 2002, 117 (1), pp. 45–58.
37. Lee, J. B. and Lee, K. B. *Lefschetz numbers for continuous maps and periods for expanding maps on infra-nilmanifolds.* Preprint.
38. Lee, K. B. *Maps of infra-nilmanifolds.* Pacific J. Math., 1995, 168, No. 1, pp. 157–166.
39. Lee, K. B. and Raymond, F. *Rigidity of almost crystallographic groups.* Contemporary Math. A. M. S., 1985, 44, pp. 73–78.

40. Mal'cev, A. I. *On a class of homogeneous spaces.* Translations A.M.S., 1951, 39, pp. 1–33.
41. Malfait, W. *Flat manifolds with prescribed first Betti number admitting Anosov diffeomorphisms.* Monatsh. Math., 2001, 133 157–162.
42. Malfait, W. *The Nielsen numbers of virtually unipotent maps on infra-nilmanifolds.* Forum Math., 2001, 13, pp. 227–237.
43. Manning, A. *There are no new Anosov diffeomorphisms on tori.* Amer. J. Math., 1974, 96 (3), pp. 422–429.
44. McCord, C. K. *Computing Nielsen numbers.* Cont. Math. A.M.S., 1993, 152, pp. 249–267.
45. McCord, C. K. *Nielsen numbers of homotopically periodic maps on infra-solvmanifolds.* Proc. Amer. Math. Soc., 1994, 120, No. 1, pp. 311–315.
46. Miatello, R. and Rossetti, J. *Isospectral Hantzsche-Wendt manifolds.* J. Reine Angew. Math., 1999, 515 pp. 1–23.
47. Miatello, R. and Rossetti, J. *Comparison of Twisted P-Form Spectra for flat manifolds with Diagonal Holonomy.* Ann. Global Ana. Geom., 2002, 21 4, pp. 341–376.
48. Norton-Odenthal, B. *A product formula for the generalized Lefschetz number.* Ph. D. Thesis, University of Wisconsin, Madison, 1991.
49. Porteous, H. L. *Anosov diffeomorphisms of flat manifolds.* Topology, 1972, 11, pp. 307–315.
50. Raghunathan, M. S. *Discrete Subgroups of Lie Groups*, volume 68 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, 1972.
51. Rossetti, J. and Szczepański, A. *Generalized Hantzsche-Wendt flat manifolds.* to appear in Revista Matem. Iberoam.
52. Warner, F. W. *Foundations of differentiable manifolds and Lie groups.* Scott, Foresman and Company, Glenview, Illinois, 1971.
53. Wecken, F. *Fixpuntklassen,III.* Math. Ann., 1942, 118, pp. 544–577.
54. Whitehead, G. *Elements of Homotopy Theory*, volume 61 of *Graduate Texts in Math.* Springer–Verlag, 1978.
55. Wong, P. *Fixed point theory for homogeneous spaces.* Amer. J. Math., 1998, 120, pp. 23–42.